

# Area-Preserving Surface Dynamics and S. Saito's Fixed Point Formula\*

Katsunori Iwasaki and Takato Uehara

Graduate School of Mathematics, Kyushu University  
6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581 Japan<sup>†</sup>

October 3, 2007

## Abstract

We show that S. Saito's fixed point formula serves as a powerful tool for counting the number of isolated periodic points of an area-preserving surface map admitting periodic curves. His notion of periodic curves of types I and II plays a central role in our discussion. We establish a Shub-Sullivan type result on the stability of local indices under iterations of the map, the finiteness of the number of periodic curves of type II, and the absence of periodic curves of type I. Combined with these results, Saito's formula implies the existence of infinitely many isolated periodic points whose cardinality grows exponentially as period tends to infinity.

## 1 Introduction

Counting the number of periodic points of a continuous map  $f : X \rightarrow X$  on a compact manifold  $X$  is an important issue in the field of dynamical systems. An efficient method for dealing with this problem is provided by several versions of Lefschetz fixed point formula. In general this formula aims at representing the Lefschetz number

$$L(f) := \sum_i (-1)^i \operatorname{Tr}[f^* : H^i(X) \rightarrow H^i(X)] \quad (1)$$

in terms of some local data around the fixed points of  $f$ . In the simplest case where all the fixed points of  $f$  are isolated in  $X$ , the classical Lefschetz formula states that

$$L(f) = \sum_{x \in X_0(f)} \nu_x(f), \quad (2)$$

where  $X_0(f)$  is the set of all fixed points of  $f$  and  $\nu_x(f)$  is the local index of  $f$  at  $x \in X_0(f)$ .

In applying formula (2) to counting periodic points, it is important to investigate the behavior of the indices  $\nu_x(f^n)$  as  $n$  tends to infinity, where  $f^n := f \circ \cdots \circ f$  ( $n$ -times) stands for the  $n$ -th iterate of  $f$ . In this regard, Shub and Sullivan [15] prove the following theorem.

---

\*Mathematics Subject Classification: 37C25, 14E07, arXiv: 0710.0706 [math.DS] 3 Oct 2007

<sup>†</sup>E-mail addresses: iwasaki@math.kyushu-u.ac.jp and t-uehara@math.kyushu-u.ac.jp

**Theorem 1.1** *Let  $X$  be a smooth manifold and  $f : X \rightarrow X$  a  $C^1$ -map. If  $x \in X$  is an isolated fixed point of all iterates  $f^n$ , then the indices  $\nu_x(f^n)$  are bounded as a function of  $n \in \mathbb{N}$ .*

Combining this theorem with the Lefschetz formula (2), they also show that if the Lefschetz numbers  $L(f^n)$  are unbounded then the set of periodic points of  $f$  is infinite. The stability result as in Theorem 1.1 is very useful for various problems in dynamical systems, so that the short paper [15] is often cited in the literature (for example in [3, 8, 7, 17]).

Unfortunately, the condition that the fixed points of a given map should be isolated is too restrictive, because one often encounters a map having a higher dimensional fixed point set. To cover this situation, many authors have made various generalizations of the classical Lefschetz formula (2) and the resulting formulas have had many fruitful applications. We refer to the pioneering work of Atiyah and Bott [1] and some subsequent works [12, 13, 16] to cite only a few. In these generalizations, however, we have to assume that the induced linear map on the normal bundle to the fixed point set should not have eigenvalue 1. This condition sometimes puts a severe restriction on the applicability of the formulas. A typical example is the case of an area-preserving map of a surface. Let  $X$  be a surface endowed with an area form and  $f : X \rightarrow X$  an area-preserving map. If  $f$  admits a fixed curve  $C$ , then  $f$  induces the identity map on the normal bundle to  $C$  so that the above-mentioned generalizations do not apply to this map (see Remark 6.3 for a more precise discussion). This situation is quite unpleasant because area-preserving maps of surfaces constitute an important class of dynamical systems. Thus there should be a more suitable fixed point formula and also a suitable variation of the Shub-Sullivan theorem which fit into this class of maps. The aim of this article is to discuss these issues upon restricting our attention to algebraic surface maps over  $\mathbb{C}$ .

At this stage we notice that S. Saito's fixed point formula [14] is very appropriate for our purpose. It is the most desirable formula that is valid for any algebraic map  $f : X \rightarrow X$  of a smooth projective surface  $X$ , where no restriction is put on the induced linear map on the normal bundle to the fixed point set. The success of his formula is due to the idea that the set  $X_1(f)$  of all irreducible fixed curves of  $f$  can be divided into two disjoint subsets, namely, into what he calls the curves of type I and those of type II:

$$X_1(f) = X_I(f) \amalg X_{II}(f). \quad (3)$$

Then his formula expresses the Lefschetz number (1) in terms of suitably defined local indices  $\nu_x(f)$  and  $\nu_C(f)$  at the fixed points  $x \in X_0(f)$  and at the fixed curves  $C \in X_1(f)$ , where different types of curves contribute to the formula in different ways; see formula (5) below.

Now we review Saito's formula more explicitly. Although his original formula in [14] is stated for holomorphic maps, we restate it for birational maps. By this alteration the resulting formula gains a wider applicability in complex dynamics, while its proof remains almost the same. Since a birational map  $f$  admits the indeterminacy set  $I(f)$  at which  $f$  is not defined, we have to restart with giving a proper definition of  $X_0(f)$  and  $X_1(f)$ . Further we have to adapt the original definitions in [14] of  $X_I(f)$ ,  $X_{II}(f)$ ,  $\nu_x(f)$  and  $\nu_C(f)$  to the current setting. Leaving all these tasks in Section 3, we now accept that these concepts are defined properly. Recall also that the induced action on cohomology and so the Lefschetz number (1) are well defined for birational maps. Now a birational version of Saito's fixed point formula is stated as follows.

**Theorem 1.2** *Let  $X$  be a smooth projective surface and  $f : X \rightarrow X$  a birational map different from the identity map. If the map  $f$  satisfies the separation condition*

$$I(f) \cap I(f^{-1}) = \emptyset, \quad (4)$$

*then the Lefschetz number of  $f$  is expressed as*

$$L(f) = \sum_{x \in X_0(f)} \nu_x(f) + \sum_{C \in X_I(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{II}(f)} \tau_C \cdot \nu_C(f), \quad (5)$$

*where  $\chi_C$  is the Euler characteristic of the normalization of  $C \in X_I(f)$  and  $\tau_C$  is the self-intersection number of  $C \in X_{II}(f)$ .*

It turns out that if the induced linear map on the normal bundle to  $C \in X_I(f)$  does not have eigenvalue 1, then  $C$  must be a fixed curve of type I (see Remark 6.3). This clearly explains why the usual generalizations of Lefschetz fixed point formula are not sufficient — they do not apply to the case where  $f$  admits fixed curves of type II. On the other hand, formula (5) is always valid, no matter which type of fixed curves are present. More remarkably, we shall see in this paper that the presence of curves of type II plays a very crucial role in discussing a Shub-Sullivan type theorem and other related issues in our context. In this sense Saito's formula (5) is a very suitable fixed point formula for our purpose.

## 2 Main Results

With the powerful fixed point formula (5) in hand, we proceed to stating the main results of this article. In what follows, unless otherwise stated explicitly,  $f : X \rightarrow X$  is a nontrivial birational map of a smooth projective surface  $X$ , where  $f$  is said to be nontrivial if it is of infinite order. In stating our main results, we do not assume the separation condition (4), which is only required for stating Theorem 1.2.

Our first main theorem is a Shub-Sullivan type result in our context. It concerns, however, the invariance of local indices rather than their boundedness as in Theorem 1.1.

**Theorem 2.1** *For any fixed curve  $C \in X_{II}(f)$  of type II, the indices  $\nu_C(f^n)$  are independent of  $n \in \mathbb{N}$ . Similarly, for any fixed point  $x \in X_0(f)$  through which at least one fixed curve of type II passes, the indices  $\nu_x(f^n)$  are independent of  $n \in \mathbb{N}$ .*

**Remark 2.2** If  $C$  is a fixed curve of type I, then  $\nu_C(f^n)$  may depend on  $n \in \mathbb{N}$ . Similarly, if a fixed point  $x$  lies outside any fixed curve of type II, then  $\nu_x(f^n)$  may depend on  $n \in \mathbb{N}$ . See Remarks 4.4 and 4.5 for some examples illustrating these remarks. The reason why the invariance of indices is more relevant than their boundedness in our context is also stated there.

Theorem 2.1 shows a special role played by curves of type II. The next main theorem exhibits another role played by these curves. To state it we introduce a bit of terminology.

**Definition 2.3** An irreducible curve  $C$  is called a *periodic curve* of  $f$  if  $C \in X_1(f^n)$  for some  $n \in \mathbb{N}$ . It is said to be of *prime period*  $n$  if  $C \in X_1(f^n)$  but  $C \notin X_1(f^m)$  for every  $m < n$ . A periodic curve  $C$  of prime period  $n$  is said to be of *type I* or of *type II* according as  $C \in X_I(f^n)$  or  $C \in X_{II}(f^n)$ , where the definition of  $X_I(f)$  and  $X_{II}(f)$  is given later in Definition 3.7.

We recall some concepts from bimeromorphic surface dynamics. Given a bimeromorphic map  $f : X \rightarrow X$  on a compact Kähler surface  $X$ , its first dynamical degree  $\lambda(f)$  is defined by

$$\lambda(f) := \lim_{n \rightarrow \infty} \|(f^n)^*|_{H^{1,1}(X)}\|^{1/n} \geq 1, \quad (6)$$

where  $\|\cdot\|$  is an operator norm on  $\text{End } H^{1,1}(X)$ . It is known that the limit exists,  $\lambda(f)$  is independent of the norm  $\|\cdot\|$  chosen and invariant under bimeromorphic conjugation (see [5]). The smallest possible value  $\lambda(f) = 1$  corresponds to the case of low dynamical complexity and so we are more interested in the case  $\lambda(f) > 1$  of higher dynamical complexity. A bimeromorphic map  $f$  is said to be *algebraically stable* (AS for short) if the condition  $(f^n)^* = (f^*)^n : H^{1,1}(X) \rightarrow H^{1,1}(X)$  holds for every  $n \in \mathbb{N}$ . It is a standard condition under which bimeromorphic surface dynamics is often discussed. If  $f$  is AS, then the first dynamical degree (6) coincides with the spectral radius of the map  $f^* : H^{1,1}(X) \rightarrow H^{1,1}(X)$ . It is known that  $f$  is AS if and only if

$$f^{-m}I(f) \cap f^n I(f^{-1}) = \emptyset \quad \text{for every } m, n \geq 0. \quad (7)$$

Now the second main theorem is concerned with the finiteness of the number of periodic curves of type II and also with the question: what happens if  $f$  has ‘too many’ periodic curves of type II?

**Theorem 2.4** *Let  $f : X \rightarrow X$  be an AS bimeromorphic map of a compact Kähler surface  $X$ .*

- (1) *If  $\lambda(f) > 1$ , then  $f$  has at most  $\rho(X) + 1$  irreducible periodic curves of type II with mutually distinct prime periods, where  $\rho(X)$  is the Picard number of  $X$ .*
- (2) *If  $\lambda(f) = 1$  and  $f^n$  is not isotopic to the identity for any  $n \in \mathbb{N}$ , then  $f$  preserves a unique rational or elliptic fibration  $\pi : X \rightarrow S$ . In addition, if  $f$  has more than  $\rho(X) + 1$  irreducible periodic curves of type II with mutually distinct prime periods, then any irreducible periodic curve of type II is contained in a fiber of the fibration  $\pi$ .*

**Remark 2.5** Two remarks are in order regarding Theorem 2.4.

- (1) Assume that  $f$  is nontrivial. Then  $f$  has at most finitely many irreducible periodic curves of type II with mutually distinct prime periods if and only if the number of all irreducible periodic curves of type II is finite (see also Remark 7.1).
- (2) If the Kodaira dimension of  $X$  is nonnegative, then the bound  $\rho(X) + 1$  in assertion (1) of Theorem 2.4 can be replaced by  $\rho(X)$ . See the proof of Theorem 2.4 in Section 5.

Theorem 2.4 asserts that the criterion  $\lambda(f) > 1$  for a high dynamical complexity implies the finiteness of periodic curves of type II, with an explicit bound  $\rho(X) + 1$  given only in terms of the geometry of  $X$ . Also in the case  $\lambda(f) = 1$  of low dynamical complexity, the presence of ‘too many’ periodic curves of type II beyond the same bound implies an even simpler dynamical behavior of  $f$ , meaning that any periodic curve of type II is along the fibration. A key ingredient to establish Theorem 2.4 is the fact that two periodic curves of type II with distinct prime periods must be disjoint (see Theorem 5.1). The classification of AS bimeromorphic surface maps due to Diller and Favre [5] is also an important ingredient.

The third main theorem is concerned with the intimate relationship between the area-preserving property of a map  $f$  and the absence of fixed curves of type I. Here we interpret the area-preserving property in a wide sense to the effect that the area form  $\omega$  preserved by  $f$  may have poles or zeros, though of course  $\omega$  must be nontrivial.

**Theorem 2.6** *Assume that  $f : X \rightarrow X$  preserves a nontrivial meromorphic 2-form  $\omega$ . If  $C$  is a fixed curve along which  $\omega$  has no pole of order  $\nu_C(f)$ , then  $C$  must be of type II. In particular any fixed curve is of type II unless it is an irreducible component of the pole divisor  $(\omega)_\infty$  of  $\omega$ .*

**Remark 2.7** If  $C$  is an irreducible component of  $(\omega)_\infty$  along which  $\omega$  has a pole of order  $\nu_C(f)$ , then  $C$  can be a fixed curve of type I. We refer to Remark 6.2 for such an example.

Looking back on what we have stated, we notice that the principal concept underlying all the above theorems is the curves of type II. Combined with these theorems, Saito's fixed point formula (5) is applied to the iterates of an area-preserving AS birational map to yield a useful formula for the number of isolated periodic points of it (see Theorem 7.2). This formula has an interesting implication as mentioned below. To state it, let  $\text{Per}_n^i(f)$  be the set of all isolated periodic points of  $f$  with (not necessarily prime) period  $n$  and  $\#\text{Per}_n^i(f)$  its cardinality counted with multiplicity (see (33) for the precise definition).

**Theorem 2.8** *Let  $f : X \rightarrow X$  be an AS birational map of a smooth projective surface  $X$  and assume that  $f$  preserves a nontrivial meromorphic 2-form  $\omega$  such that*

*(\*) no irreducible component of the pole divisor  $(\omega)_\infty$  of  $\omega$  is a periodic curve of type I.*

*If  $\lambda(f) > 1$  then  $f$  has at most finitely many irreducible periodic curves and infinitely many isolated periodic points. Moreover the number of isolated periodic points of period  $n$ , counted with multiplicity, is estimated as*

$$|\#\text{Per}_n^i(f) - \lambda(f)^n| \leq \begin{cases} O(1) & (\text{if } X \sim \text{no Abelian surface}), \\ 4\lambda(f)^{n/2} + O(1) & (\text{if } X \sim \text{an Abelian surface}), \end{cases}$$

where  $O(1)$  stands for a bounded function of  $n \in \mathbb{N}$  and  $X \sim Y$  indicates that  $X$  is birationally equivalent to  $Y$ .

In this sense, area-preserving surface maps nicely fit into Saito's fixed point formula (5) and hence the title of this article. Note that, once  $f$  and  $\omega$  are given concretely, the condition (\*) is verifiable in finite procedures, since  $(\omega)_\infty$  contains only finitely many irreducible components.

The plan of this article is as follows. After a more detailed review of Saito's fixed point formula in Section 3, Theorems 2.1, 2.4, 2.6 and 2.8 are proved in Sections 4, 5, 6 and 7 respectively. Actually a refined version of Theorem 2.8 is established in Section 7 (see Theorem 7.9). In Section 8 we illustrate our main theorems by giving an interesting example of an area-preserving AS map on the minimal resolution of a singular cubic surface.

### 3 S. Saito's Fixed Point Formula

In this section we introduce the terminology and concepts needed to formulate Theorem 1.2, the birational version of S. Saito's fixed point formula in [14]. These terminology and concepts are also needed to formulate our main theorems in Section 2.

We begin with some ring-theoretical preparations. Let  $A := \mathbb{C}[[z_1, z_2]]$  be the ring of formal power series of two variables over  $\mathbb{C}$  with its maximal ideal  $\mathfrak{m} \subset A$  and let  $\sigma : A \rightarrow A$  be a nontrivial continuous endomorphism of  $A$  in the  $\mathfrak{m}$ -adic topology. Then  $\sigma$  is expressed as

$$\begin{cases} \sigma(z_1) &= z_1 + g \cdot h_1, \\ \sigma(z_2) &= z_2 + g \cdot h_2, \end{cases} \quad (8)$$

for some elements  $g, h_1, h_2 \in A$ , where  $g$  is nonzero and  $h_1, h_2$  are relatively prime. Consider the ideals  $\mathfrak{a}(\sigma) := (g)$  and  $\mathfrak{b}(\sigma) := (h_1, h_2)$  generated by  $g$  and by  $h_1, h_2$ , respectively. They are independent of the choice of the coordinates  $z_1$  and  $z_2$ . Since  $h_1$  and  $h_2$  are relatively prime, the quotient vector space  $A/\mathfrak{b}(\sigma)$  is finite-dimensional, so that one can put

$$\delta(\sigma) := \dim_{\mathbb{C}} A/\mathfrak{b}(\sigma) < \infty. \quad (9)$$

Let  $\Lambda(\sigma)$  be the set of all prime ideals  $\mathfrak{p}$  of height 1 in  $A$  that divide  $\mathfrak{a}(\sigma)$ . For  $\mathfrak{p} \in \Lambda(\sigma)$ , put

$$\nu_{\mathfrak{p}}(\sigma) := \max\{m \in \mathbb{N} \mid \mathfrak{a}(\sigma) \subset \mathfrak{p}^m\}. \quad (10)$$

Let  $\kappa[\mathfrak{p}]$  be the normalization of the quotient ring  $A/\mathfrak{p}$ , and  $\kappa(\mathfrak{p})$  the quotient field of  $\kappa[\mathfrak{p}]$ . It follows from the definition that  $\kappa[\mathfrak{p}]$  is isomorphic to  $\mathbb{C}[[t]]$  for some prime element  $t$ . Moreover, two modules of formal differentials are defined by the projective limits:

$$\hat{\Omega}_{A/\mathbb{C}}^1 := \varprojlim_n \Omega_{A_n/\mathbb{C}}^1, \quad \hat{\Omega}_{\kappa[\mathfrak{p}]/\mathbb{C}}^1 := \varprojlim_n \Omega_{\kappa[\mathfrak{p}]_n/\mathbb{C}}^1,$$

where  $A_n := A/\mathfrak{m}^n$  and  $\kappa[\mathfrak{p}]_n := \mathbb{C}[[t]]/(t)^n$ . It is easy to see that  $\hat{\Omega}_{A/\mathbb{C}}^1$  is a free  $A$ -module of rank two with generators  $dz_1$  and  $dz_2$ , while  $\hat{\Omega}_{\kappa[\mathfrak{p}]/\mathbb{C}}^1$  is a free  $\kappa[\mathfrak{p}]$ -module of rank one with a generator  $dt$ . Furthermore we put  $\hat{\Omega}_{\kappa(\mathfrak{p})/\mathbb{C}}^1 := \hat{\Omega}_{\kappa[\mathfrak{p}]/\mathbb{C}}^1 \otimes_{\kappa[\mathfrak{p}]} \kappa(\mathfrak{p})$  and define a map

$$\tau_{\mathfrak{p}} : \hat{\Omega}_{A/\mathbb{C}}^1 \rightarrow \hat{\Omega}_{\kappa(\mathfrak{p})/\mathbb{C}}^1$$

to be the homomorphism induced from the natural map  $A \rightarrow \kappa(\mathfrak{p})$ . Finally we put

$$\varpi_{\sigma} := h_2 \cdot dz_1 - h_1 \cdot dz_2 \in \hat{\Omega}_{A/\mathbb{C}}^1. \quad (11)$$

**Definition 3.1** A prime ideal  $\mathfrak{p} \in \Lambda(\sigma)$  is said to be of *type I* relative to  $\sigma$ , if  $\tau_{\mathfrak{p}}(\varpi_{\sigma}) \neq 0$  in  $\hat{\Omega}_{\kappa(\mathfrak{p})/\mathbb{C}}^1$ ; otherwise,  $\mathfrak{p}$  is said to be of *type II*. Let  $\Lambda_I(\sigma)$  denote the set of prime ideals  $\mathfrak{p} \in \Lambda(\sigma)$  of type I and let  $\Lambda_{II}(\sigma)$  denote the set of prime ideals  $\mathfrak{p} \in \Lambda(\sigma)$  of type II, respectively.

In view of this definition we observe that there exists an element  $a \in \kappa[\mathfrak{p}]$  such that

$$\begin{cases} \tau_{\mathfrak{p}}(\varpi_{\sigma}) = a \cdot dt & (\text{if } \mathfrak{p} \in \Lambda_I(\sigma)), \\ \varpi_{\sigma} = a \cdot dp \pmod{\mathfrak{p} \cdot \hat{\Omega}_{A/\mathbb{C}}^1} & (\text{if } \mathfrak{p} \in \Lambda_{II}(\sigma)), \end{cases} \quad (12)$$

where  $p \in A$  is a prime element such that  $\mathfrak{p} = (p)$ . Identifying  $\kappa[\mathfrak{p}]$  with  $\mathbb{C}[[t]]$ , we define

$$\mu_{\mathfrak{p}}(\sigma) := \max\{m \in \mathbb{N} \cup \{0\} \mid (a) \subset (t)^m\} \quad (\mathfrak{p} \in \Lambda(\sigma)) \quad (13)$$

$$\nu_A(\sigma) := \delta(\sigma) + \sum_{\mathfrak{p} \in \Lambda(\sigma)} \nu_{\mathfrak{p}}(\sigma) \cdot \mu_{\mathfrak{p}}(\sigma). \quad (14)$$

The case where  $\sigma$  is an automorphism will be of particular interest later.

**Lemma 3.2** *If  $\sigma$  is an automorphism, then we have:*

- $\Lambda(\sigma^{-1}) = \Lambda(\sigma)$ ,  $\Lambda_I(\sigma^{-1}) = \Lambda_I(\sigma)$  and  $\Lambda_{II}(\sigma^{-1}) = \Lambda_{II}(\sigma)$ ;
- $\delta(\sigma^{-1}) = \delta(\sigma)$  and  $\nu_A(\sigma^{-1}) = \nu_A(\sigma)$ ;
- $\nu_{\mathfrak{p}}(\sigma^{-1}) = \nu_{\mathfrak{p}}(\sigma)$  and  $\mu_{\mathfrak{p}}(\sigma^{-1}) = \mu_{\mathfrak{p}}(\sigma)$  for any  $\mathfrak{p} \in \Lambda(\sigma)$ .

*Proof.* As in (8), the inverse automorphism  $\sigma^{-1} : A \rightarrow A$  is expressed as

$$\begin{cases} \sigma^{-1}(z_1) &= z_1 + u \cdot v_1, \\ \sigma^{-1}(z_2) &= z_2 + u \cdot v_2, \end{cases}$$

for some  $u, v_1, v_2 \in A$ , where  $u \neq 0$  and  $v_1, v_2$  are relatively prime. Applying  $\sigma$  to it, we have

$$\begin{cases} \sigma(z_1) = z_1 - \sigma(u) \cdot \sigma(v_1), \\ \sigma(z_2) = z_2 - \sigma(u) \cdot \sigma(v_2), \end{cases}$$

where  $\sigma(v_1)$  and  $\sigma(v_2)$  are relatively prime, since so are  $v_1$  and  $v_2$ , and  $\sigma$  is an automorphism. Comparing this with (8) and multiplying  $g$  by a unit and  $h_i$  by its inverse, we can put  $-g = \sigma(u)$  and  $h_i = \sigma(v_i)$  for  $i = 1, 2$ . Expanding the righthand sides into a Taylor series yields

$$\begin{cases} -g = \sigma(u) = u(z_1 + g \cdot h_1, z_2 + g \cdot h_2) = u + g \cdot b, \\ h_i = \sigma(v_i) = v_i(z_1 + g \cdot h_1, z_2 + g \cdot h_2) = v_i + g \cdot b_i \quad (i = 1, 2), \end{cases} \quad (15)$$

with some elements  $b, b_i \in \mathfrak{b}(\sigma)$ . Hence  $u = -g(1 + b) \in \mathfrak{a}(\sigma)$  and  $v_i = h_i - g \cdot b_i \in \mathfrak{b}(\sigma)$  for  $i = 1, 2$ , so that  $\mathfrak{a}(\sigma^{-1}) \subset \mathfrak{a}(\sigma)$  and  $\mathfrak{b}(\sigma^{-1}) \subset \mathfrak{b}(\sigma)$ . Replacing  $\sigma$  with  $\sigma^{-1}$  we also have  $\mathfrak{a}(\sigma) \subset \mathfrak{a}(\sigma^{-1})$  and  $\mathfrak{b}(\sigma) \subset \mathfrak{b}(\sigma^{-1})$ . Thus  $\mathfrak{a}(\sigma^{-1}) = \mathfrak{a}(\sigma)$  and  $\mathfrak{b}(\sigma^{-1}) = \mathfrak{b}(\sigma)$ . In view of (9), (10) and the definition of  $\Lambda(\sigma)$ , these equalities imply that  $\Lambda(\sigma^{-1}) = \Lambda(\sigma)$ ,  $\delta(\sigma^{-1}) = \delta(\sigma)$  and  $\nu_{\mathfrak{p}}(\sigma^{-1}) = \nu_{\mathfrak{p}}(\sigma)$  for any  $\mathfrak{p} \in \Lambda(\sigma)$ . Moreover it follows from (11) and (15) that

$$\varpi_{\sigma} = h_2 \cdot dz_1 - h_1 \cdot dz_2 = (v_2 \cdot dz_1 - v_1 \cdot dz_2) + g(b_2 \cdot dz_1 - b_1 \cdot dz_2) \in \varpi_{\sigma^{-1}} + \mathfrak{a}(\sigma) \cdot \hat{\Omega}_{A/\mathbb{C}}^1.$$

In view of (12) and (13), this shows that  $\Lambda_I(\sigma^{-1}) = \Lambda_I(\sigma)$ ,  $\Lambda_{II}(\sigma^{-1}) = \Lambda_{II}(\sigma)$  and  $\mu_{\mathfrak{p}}(\sigma^{-1}) = \mu_{\mathfrak{p}}(\sigma)$  for any  $\mathfrak{p} \in \Lambda(\sigma)$ . Finally,  $\nu_A(\sigma^{-1}) = \nu_A(\sigma)$  readily follows from (14). ■

We turn our attention to birational surface maps. Let  $f : X \rightarrow X$  be a nontrivial birational map of a smooth projective surface  $X$ . Since  $f$  admits the indeterminacy set  $I(f)$  at which  $f$  is not defined, we must ask what should be the definition of  $X_0(f)$ , the set of fixed points of  $f$ . A natural idea is to treat the forward map  $f$  and the backward map  $f^{-1}$  symmetrically so that one can switch between  $f$  and  $f^{-1}$ . Then even for a point  $x \in I(f)$  one can declare that  $x$  is a fixed point of  $f$  provided that  $x$  is away from  $I(f^{-1})$  and is fixed by  $f^{-1}$ . The definition of  $X_1(f)$  also needs some care, though it is just a small matter.

**Definition 3.3** Let  $X_0^{\circ}(f)$  be the set of all points  $x \in X \setminus I(f)$  fixed by  $f$  and put

$$X_0(f) := X_0^{\circ}(f) \cup X_0^{\circ}(f^{-1}). \quad (16)$$

Let  $X_1(f)$  be the set of all irreducible curves  $C$  in  $X$  such that  $C \setminus I(f)$  is fixed pointwise by  $f$ . This definition makes sense since  $I(f)$  is a finite set of points and so  $C \setminus I(f)$  is a nonempty Zariski open subset of  $C$ . It is easy to see that the definition is symmetric:

$$X_1(f) = X_1(f^{-1}). \quad (17)$$

**Remark 3.4** Let  $E(f)$  be the exceptional set of  $f$ . Any irreducible component  $C$  of  $E(f)$  is not an element of  $X_1(f)$ , because  $C \setminus I(f)$  is contracted to a single point by  $f$ .

We now define the index  $\nu_x(f)$  at a fixed point  $x \in X_0(f)$  and the index  $\nu_C(f)$  at a fixed curve  $C \in X_1(f)$ . Let  $A_x$  denote the completion of the local ring of  $X$  at  $x$ . If  $x \in X_0^\circ(f)$  then the map  $f$  is holomorphic around  $x$  and hence induces a continuous endomorphism  $f_x^* : A_x \rightarrow A_x$  with respect to the  $\mathfrak{m}_x$ -adic topology in a natural manner, where  $\mathfrak{m}_x$  is the maximal ideal of  $A_x$ . Since  $X$  is assumed to be smooth, the ring  $A_x$  is isomorphic to the formal power series ring  $\mathbb{C}[[z_1, z_2]]$ , so that upon putting  $A = A_x$  and  $\sigma = f_x^*$  we can come to the ring-theoretical situation considered above and define the number  $\nu_{A_x}(f_x^*)$  via the formula (14). Similarly, if  $x \in X_0^\circ(f^{-1})$  then we can consider the number  $\nu_{A_x}((f^{-1})_x^*)$  instead of  $\nu_{A_x}(f_x^*)$ . Moreover, if  $x \in X_0^\circ(f) \cap X_0^\circ(f^{-1})$ , then  $f$  is a local biholomorphism around  $x$ , inducing an automorphism  $f_x^* : A_x \rightarrow A_x$  with its inverse  $(f_x^*)^{-1} = (f^{-1})_x^*$ , so that Lemma 3.2 implies that

$$\nu_{A_x}(f_x^*) = \nu_{A_x}((f^{-1})_x^*) \quad \text{at } x \in X_0^\circ(f) \cap X_0^\circ(f^{-1}). \quad (18)$$

Next, given a fixed curve  $C \in X_1(f)$ , take a point  $x$  of  $C \setminus I(f)$ . Then one can speak of the continuous endomorphism  $f_x^* : A_x \rightarrow A_x$ . Let  $C_x$  denote the germ at  $x$  of the curve  $C$  and let  $\Lambda(C_x)$  be the set of all prime ideals in  $A_x$  determined by the irreducible components of  $C_x$ . Then any  $\mathfrak{p} \in \Lambda(C_x)$  is a prime ideal of length 1 that divides  $\mathfrak{a}(f_x^*)$ , that is,  $\Lambda(C_x) \subset \Lambda(f_x^*)$ , so that one can define the number  $\nu_{\mathfrak{p}}(f_x^*)$  via the formula (10) with  $\sigma = f_x^*$ . This definition does not depend on the choice of the point  $x \in C \setminus I(f)$  and the ideal  $\mathfrak{p} \in \Lambda(C_x)$  (see Saito [14, page 1016]). Summing up these discussions, we make the following definitions.

**Definition 3.5** The local index  $\nu_x(f)$  at a fixed point  $x \in X_0(f)$  is defined by

$$\nu_x(f) := \begin{cases} \nu_{A_x}(f_x^*) & (\text{if } x \in X_0^\circ(f)), \\ \nu_{A_x}((f^{-1})_x^*) & (\text{if } x \in X_0^\circ(f^{-1})), \end{cases} \quad (19)$$

which is consistent by virtue of (18). The index  $\nu_C(f)$  at a fixed curve  $C \in X_1(f)$  is defined by

$$\nu_C(f) := \nu_{\mathfrak{p}}(f_x^*) \quad (20)$$

with a (any) point  $x \in C \setminus I(f)$  and an (any) prime ideal  $\mathfrak{p} \in \Lambda(C_x)$ .

**Remark 3.6** We have  $\nu_x(f) > 0$  for at most finitely many points  $x \in X_0(f)$ .

**Definition 3.7** A fixed curve  $C \in X_1(f)$  is said to be of *type I* or of *type II* relative to  $f : X \rightarrow X$  according as the prime ideal  $\mathfrak{p} \in \Lambda(C_x)$  is of type I or of type II relative to  $f_x^* : A_x \rightarrow A_x$  in the sense of Definition 3.1. This definition does not depend on the choice of the point  $x \in C \setminus I(f)$  and the ideal  $\mathfrak{p} \in \Lambda(C_x)$  (see [14, page 1016]). Let  $X_I(f)$  and  $X_{II}(f)$  denote the set of fixed curves of types I and the set of fixed curves of type II respectively. Then there exists the direct sum decomposition as in (3).

The preparation of all terminology and concepts needed to formulate Theorem 1.2 is now complete. The separation condition (4) means that if  $x \in X$  is an indeterminacy point of  $f$  then  $x$  is a holomorphic point of  $f^{-1}$  and vice versa. Thus, under condition (4), all the fixed points needed to validate formula (5) are captured by the union  $X_0(f) := X_0^\circ(f) \cup X_0^\circ(f^{-1})$ . This is



why we make the definition (16). The consistency (18) with respect to  $f^{\pm 1}$  is well understood by the symmetry of the graphs  $\Gamma_{f^{\pm 1}}$  of  $f^{\pm 1}$ , that is, by the fact that one graph is the reflection of the other in the diagonal  $\Delta$  of  $X \times X$ . This symmetry indicates the naturality of definition (19) since  $\nu_{Ax}((f^{\pm 1})_x^*)$  represent the degrees of intersection between  $\Gamma_{f^{\pm 1}}$  and  $\Delta$  at the point  $(x, x)$ . It is in these settings that Theorem 1.2 is valid. A remark is in order at this stage.

**Remark 3.8** In dynamical situations the fixed point formula (5) is to be applied to the iterates  $f^n$  of a map  $f$ , so that the separation condition (4) should be replaced by its iterated version:

$$I(f^n) \cap I(f^{-n}) = \emptyset \quad \text{for every } n \in \mathbb{N}.$$

It is easy to see that this condition follows from the AS condition (7). So the AS birational maps constitute a nice class of maps to which the fixed point formula (5) can be applied dynamically.

## 4 Stability of Indices

The goal of this section is to establish Theorem 2.1. This boils down to showing the following theorem in the abstract ring-theoretical setting as in the first part of Section 3.

**Theorem 4.1** *Let  $A := \mathbb{C}[[z_1, z_2]]$  and  $\sigma : A \rightarrow A$  a nontrivial continuous endomorphism in the  $\mathfrak{m}$ -adic topology. If  $\Lambda_{II}(\sigma)$  is nonempty, then for any  $n \in \mathbb{N}$ ,*

- $\Lambda(\sigma^n) = \Lambda(\sigma)$ ,  $\Lambda_I(\sigma^n) = \Lambda_I(\sigma)$  and  $\Lambda_{II}(\sigma^n) = \Lambda_{II}(\sigma)$ ;
- $\delta(\sigma^n) = \delta(\sigma)$  and  $\nu_A(\sigma^n) = \nu_A(\sigma)$ ;
- $\nu_{\mathfrak{p}}(\sigma^n) = \nu_{\mathfrak{p}}(\sigma)$  and  $\mu_{\mathfrak{p}}(\sigma^n) = \mu_{\mathfrak{p}}(\sigma)$  for any  $\mathfrak{p} \in \Lambda(\sigma)$ .

In order to prove this theorem we need some preliminaries. As in (8), for each  $n \in \mathbb{N}$  the endomorphism  $\sigma^n : A \rightarrow A$  can be expressed as

$$\begin{cases} \sigma^n(z_1) &= z_1 + g_n \cdot h_{n1}, \\ \sigma^n(z_2) &= z_2 + g_n \cdot h_{n2}, \end{cases} \quad (21)$$

for some elements  $g_n, h_{n1}, h_{n2} \in A$ , where  $g_n$  is nonzero and  $h_{n1}, h_{n2}$  are relatively prime. By definition  $\mathfrak{a}(\sigma^n) := (g_n)$  and  $\mathfrak{b}(\sigma^n) := (h_{n1}, h_{n2})$  are the ideals generated by  $g_n$  and by  $h_{n1}, h_{n2}$ , respectively. To simplify the notation we put  $g := g_1$  and  $h_i := h_{1i}$ .

**Lemma 4.2** *If  $\Lambda_{II}(\sigma)$  is nonempty, then for any  $n \in \mathbb{N}$  and  $\mathfrak{p} \in \Lambda_{II}(\sigma)$  we have:*

$$\mathfrak{a}(\sigma^n) = \mathfrak{a}(\sigma), \quad \mathfrak{b}(\sigma^n) = \mathfrak{b}(\sigma), \quad h_{ni} \in n \cdot h_i + \mathfrak{p} \cdot \mathfrak{b}(\sigma).$$

*Proof.* We prove the lemma by induction on  $n \in \mathbb{N}$ . It is trivial for  $n = 1$ . Assume that the lemma holds for  $n \in \mathbb{N}$ . In what follows, to make the presentation simpler, we use the symbol  $b_i$  to denote various elements of the ideal  $\mathfrak{b}(\sigma) = (h_1, h_2)$ . This abuse of notation causes no confusion when we are only interested in the argument modulo  $\mathfrak{b}(\sigma)$ . Let  $g_{z_i}$  denote the formal

partial derivative of  $g$  with respect to  $z_i$  ( $i = 1, 2$ ). Considering the formal Taylor expansion of  $\sigma^{n+1}(z_i) = \sigma(\sigma^n(z_i))$  with (8) and (21) taken into account, we have

$$\begin{aligned}
\sigma^{n+1}(z_i) &= \sigma(\sigma^n(z_i)) \\
&= z_i + g \cdot h_i + g_n(z_1 + g \cdot h_1, z_2 + g \cdot h_2) \cdot h_{ni}(z_1 + g \cdot h_1, z_2 + g \cdot h_2) \\
&= z_i + g \cdot h_i + g(z_1 + g \cdot h_1, z_2 + g \cdot h_2) \cdot h_{ni}(z_1 + g \cdot h_1, z_2 + g \cdot h_2) \\
&= z_i + g \cdot h_i + (g + g_{z_1} \cdot g \cdot h_1 + g_{z_2} \cdot g \cdot h_2 + g^2 \cdot b_i) \cdot (h_{ni} + g \cdot b_i) \\
&= z_i + g \cdot \{h_i + h_{ni} \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) + g \cdot b_i\} \\
&= z_i + g \cdot \tilde{h}_{ni} \quad (i = 1, 2),
\end{aligned} \tag{22}$$

where we use the induction hypothesis  $\mathbf{a}(\sigma^n) = \mathbf{a}(\sigma)$  in the third line and we put

$$\tilde{h}_{ni} := h_i + h_{ni} \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) + g \cdot b_i. \tag{23}$$

in the last line. We investigate the term  $g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2$ . Take a prime element  $p \in A$  such that  $\mathfrak{p} = (p)$ . Since  $\mathfrak{p} = (p)$  divides  $\mathbf{a}(\sigma) = (g)$ , the formal power series  $g$  can be written

$$g = p^m \cdot \tilde{g} \tag{24}$$

for some  $m \in \mathbb{N}$  and  $\tilde{g} \in A$ . Since  $\mathfrak{p} \in \Lambda_H(\sigma)$ , formula (12) implies that the formal differential  $\varpi_\sigma$  in (11) is expressed as  $\varpi_\sigma := h_2 \cdot dz_1 - h_1 \cdot dz_2 = a \cdot dp + p \cdot (q_2 \cdot dz_1 - q_1 \cdot dz_2)$  with some elements  $a, q_1, q_2 \in A$ . Comparing the coefficients of  $dz_1$  and  $dz_2$  yields

$$\begin{cases} h_1 = -a \cdot p_{z_2} + p \cdot q_1, \\ h_2 = a \cdot p_{z_1} + p \cdot q_2. \end{cases} \tag{25}$$

where  $p_{z_i}$  is the formal partial derivative of  $p$  with respect to  $z_i$ . By (24) and (25),

$$\begin{aligned}
g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2 &= g_{z_1} \cdot (-a \cdot p_{z_2} + p \cdot q_1) + g_{z_2} \cdot (a \cdot p_{z_1} + p \cdot q_2) \\
&= (m \cdot p^{m-1} \cdot p_{z_1} \cdot \tilde{g} + p^m \cdot \tilde{g}_{z_1}) \cdot (-a \cdot p_{z_2}) \\
&\quad + (m \cdot p^{m-1} \cdot p_{z_2} \cdot \tilde{g} + p^m \cdot \tilde{g}_{z_2}) \cdot (a \cdot p_{z_1}) + p \cdot (g_{z_1} \cdot q_1 + g_{z_2} \cdot q_2) \\
&= a \cdot p^m \cdot (p_{z_1} \cdot \tilde{g}_{z_2} - p_{z_2} \cdot \tilde{g}_{z_1}) + p \cdot (g_{z_1} \cdot q_1 + g_{z_2} \cdot q_2) \\
&= p \cdot \{a \cdot p^{m-1} \cdot (p_{z_1} \cdot \tilde{g}_{z_2} - p_{z_2} \cdot \tilde{g}_{z_1}) + (g_{z_1} \cdot q_1 + g_{z_2} \cdot q_2)\} \in \mathfrak{p}.
\end{aligned}$$

Using this,  $g \in \mathfrak{p}$  and the induction hypothesis  $h_{ni} \in n \cdot h_i + \mathfrak{p} \cdot \mathbf{b}(\sigma)$  in (23), we have

$$\tilde{h}_{ni}(z) \in (n+1) \cdot h_i(z) + \mathfrak{p} \cdot \mathbf{b}(\sigma) \quad (i = 1, 2). \tag{26}$$

Now consider the ideal  $\tilde{\mathbf{b}}(\sigma^n) := (\tilde{h}_{n1}, \tilde{h}_{n2})$  generated by  $\tilde{h}_{n1}$  and  $\tilde{h}_{n2}$ . We show that

$$\tilde{\mathbf{b}}(\sigma^n) = \mathbf{b}(\sigma). \tag{27}$$

The inclusion  $\tilde{\mathbf{b}}(\sigma^n) \subset \mathbf{b}(\sigma)$  is obvious, since  $\tilde{h}_{n1}, \tilde{h}_{n2} \in \mathbf{b}(\sigma)$  by (26). On the other hand, formula (26) also implies that there exist elements  $r_{nij} \in \mathfrak{p}$  ( $i, j = 1, 2$ ) such that

$$\begin{cases} \tilde{h}_{n1} &= (n+1) \cdot h_1 + r_{n11} \cdot h_1 + r_{n12} \cdot h_2 &= (n+1+r_{n11}) \cdot h_1 + r_{n12} \cdot h_2, \\ \tilde{h}_{n2} &= (n+1) \cdot h_2 + r_{n21} \cdot h_1 + r_{n22} \cdot h_2 &= r_{n21} \cdot h_1 + (n+1+r_{n22}) \cdot h_2, \end{cases}$$

If we put  $r := (n + 1 + r_{n11}) \cdot (n + 1 + r_{n22}) - r_{n12} \cdot r_{n21}$ , then these equations yield

$$\begin{cases} r \cdot h_1 &= (n + 1 + r_{n22}) \cdot \tilde{h}_{n1} - r_{n12} \cdot \tilde{h}_{n2}, \\ r \cdot h_2 &= r_{n21} \cdot \tilde{h}_{n1} - (n + 1 + r_{n11}) \cdot \tilde{h}_{n2}. \end{cases}$$

Since  $r_{nij} \in \mathfrak{m}$  ( $i, j = 1, 2$ ), the factor  $r$  is an invertible element of  $A$ , so that one has  $h_1, h_2 \in \tilde{\mathfrak{b}}(\sigma^n)$ . This yields the reverse inclusion  $\mathfrak{b}(\sigma) \subset \tilde{\mathfrak{b}}(\sigma^n)$  and the claim (27) is proved. Since  $h_1$  and  $h_2$  are relatively prime, the equality (27) implies that  $\tilde{h}_{n1}$  and  $\tilde{h}_{n2}$  are also relatively prime, so that from (22) one can conclude that

$$g_{n+1} = g, \quad h_{n+1,i} = \tilde{h}_{ni} \quad (i = 1, 2). \quad (28)$$

The first equality of (28) yields  $\mathfrak{a}(\sigma^{n+1}) = \mathfrak{a}(\sigma)$ . The second equality of (28) and (27) lead to  $\mathfrak{b}(\sigma^{n+1}) = \mathfrak{b}(\sigma)$ . Finally the second equality of (28) and (26) give  $h_{n+1,i} \in (n+1) \cdot h_i(z) + \mathfrak{p} \cdot \mathfrak{b}(\sigma)$ . Thus the induction is complete. ■

**Lemma 4.3** *If  $\Lambda_{II}(\sigma)$  is nonempty then for any  $\mathfrak{q} \in \Lambda_I(\sigma)$  there is  $c_n \in \mathfrak{m}$  such that*

$$h_{ni} \in (n + c_n) \cdot h_i + \mathfrak{q}.$$

*Proof.* We prove the lemma by induction on  $n \in \mathbb{N}$ . Let  $\mathfrak{p} \in \Lambda_{II}(\sigma)$  be a prime ideal of type II as in Lemma 4.2 and its proof. It follows from (23) and (28) that

$$h_{n+1,i} = h_i + h_{ni} \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) + g \cdot b_i \quad (i = 1, 2), \quad (29)$$

where  $b_i \in \mathfrak{b}(f^*)$ . Let  $q \in \mathfrak{m}$  be a prime element such that  $\mathfrak{q} = (q)$ . Since  $\mathfrak{q}$  is different from  $\mathfrak{p}$ , the product  $p \cdot q$  divides  $g$ . Moreover, since  $p, q \in \mathfrak{m}$ , we have  $g \in \mathfrak{q}$  and  $g \in \mathfrak{m}^2$ . If the assertion holds for  $n$ , then it follows from (29) that

$$\begin{aligned} h_{n+1,i} &= h_i + h_{ni} \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) + g \cdot b_i \\ &\equiv h_i + h_{ni} \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) \pmod{\mathfrak{q}} \\ &\equiv h_i + (n + c_n) \cdot h_i \cdot (1 + g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2) \pmod{\mathfrak{q}} \\ &= h_i \cdot \{(n + 1) + c_n + (n + c_n) \cdot (g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2)\} \\ &= h_i \cdot \{(n + 1) + c_{n+1}\}, \end{aligned}$$

where we use  $g \in \mathfrak{q}$  in the second line and the induction hypothesis in the third line, and we put  $c_{n+1} := c_n + (n + c_n) \cdot (g_{z_1} \cdot h_1 + g_{z_2} \cdot h_2)$  in the last line. Since  $g \in \mathfrak{m}^2$ , we have  $g_{z_i} \in \mathfrak{m}$  and hence  $c_{n+1} \in \mathfrak{m}$ . Thus the assertion is true for  $n + 1$  and the induction is complete. ■

*Proof of Theorem 4.1.* In view of (9), (10) and the definition of  $\Lambda(\sigma)$ , the equalities  $\mathfrak{a}(\sigma^n) = \mathfrak{a}(\sigma)$  and  $\mathfrak{b}(\sigma^n) = \mathfrak{b}(\sigma)$  in Lemma 4.2 imply that  $\Lambda(\sigma^n) = \Lambda(\sigma)$ ,  $\delta(\sigma^n) = \delta(\sigma)$  and  $\nu_{\mathfrak{p}}(\sigma^n) = \nu_{\mathfrak{p}}(\sigma)$  for any  $\mathfrak{p} \in \Lambda(\sigma)$ . If  $\mathfrak{p} \in \Lambda_{II}(\sigma)$ , then (11) and the last formula of Lemma 4.2 imply that

$$\varpi_{\sigma^n} = h_{n2} \cdot dz_1 - h_{n1} \cdot dz_2 \in n \cdot (h_2 \cdot dz_1 - h_1 \cdot dz_2) + \mathfrak{p} \cdot \hat{\Omega}_{A/\mathbb{C}}^1 = n \cdot \varpi_{\sigma} + \mathfrak{p} \cdot \hat{\Omega}_{A/\mathbb{C}}^1,$$

and so  $\tau_{\mathfrak{p}}(\varpi_{\sigma^n}) = n \cdot \tau_{\mathfrak{p}}(\varpi_{\sigma}) = 0$ , which means that  $\mathfrak{p} \in \Lambda_{II}(\sigma^n)$ . Moreover (12) and (13) yield  $\mu_{\mathfrak{p}}(\sigma^n) = \mu_{\mathfrak{p}}(\sigma)$ . Next assume that  $\mathfrak{p} \in \Lambda_I(\sigma)$  and rewrite  $\mathfrak{p} = \mathfrak{q}$ . By (11) and Lemma 4.3,

$$\begin{aligned} \varpi_{\sigma^n} = h_{n2} \cdot dz_1 - h_{n1} \cdot dz_2 &\in (n + c_n) \cdot (h_2 \cdot dz_1 - h_1 \cdot dz_2) + \mathfrak{q} \cdot \hat{\Omega}_{A/\mathbb{C}}^1 \\ &= (n + c_n) \cdot \varpi_{\sigma} + \mathfrak{q} \cdot \hat{\Omega}_{A/\mathbb{C}}^1, \end{aligned}$$

where  $n + c_n$  is an invertible element of  $A$ . Hence  $\tau_{\mathbf{q}}(\varpi_{\sigma^n}) = (n + c_n) \cdot \tau_{\mathbf{q}}(\varpi_{\sigma}) \neq 0$ , which means that  $\mathbf{q} \in \Lambda_I(\sigma^n)$ . Moreover (12) and (13) yield  $\mu_{\mathbf{q}}(\sigma^n) = \mu_{\mathbf{q}}(\sigma)$ . Therefore we have  $\Lambda_I(\sigma^n) = \Lambda_I(\sigma)$ ,  $\Lambda_{II}(\sigma^n) = \Lambda_{II}(\sigma)$  and  $\mu_{\mathbf{p}}(\sigma^n) = \mu_{\mathbf{p}}(\sigma)$  for any  $\mathbf{p} \in \Lambda(\sigma)$ . Finally the equality  $\nu_A(\sigma^n) = \nu_A(\sigma)$  readily follows from (14). The proof is complete. ■

We are now in a position to establish Theorem 2.1.

*Proof of Theorem 2.1.* Let  $C \in X_{II}(f)$  and take a point  $x \in C \setminus I(f)$ . Then for each  $n \in \mathbb{N}$  the map  $f^n$  induces an endomorphism  $(f^n)_x^* = (f_x^*)^n : A_x \rightarrow A_x$ . Take any prime ideal  $\mathbf{p} \in \Lambda(C_x)$ . Since  $C \in X_{II}(f)$ , we have  $\mathbf{p} \in \Lambda_{II}(f_x^*)$ . Hence it follows from (20) and Theorem 4.1 that

$$\nu_C(f^n) = \nu_{\mathbf{p}}((f_x^*)^n) = \nu_{\mathbf{p}}(f_x^*) = \nu_C(f),$$

which proves the first assertion of the theorem. Next, let  $x \in X_0(f)$  be a fixed point of  $f$  through which at least one fixed curve, say,  $C \in X_{II}(X)$  of type II passes. In view of (16) we may assume that  $x \in X_0^\circ(f)$ , namely, that  $x \in C \setminus I(f)$ ; for, otherwise, we can replace  $f$  by  $f^{-1}$  and proceed in a similar manner. Now the endomorphisms  $(f^n)_x^* = (f_x^*)^n : A_x \rightarrow A_x$  make sense and  $\Lambda_{II}(f_x^*)$  is nonempty. Hence (19) and Theorem 4.1 imply that

$$\nu_x(f^n) = \nu_{A_x}((f_x^*)^n) = \nu_{A_x}(f_x^*) = \nu_x(f),$$

which proves the second assertion of the theorem. Therefore Theorem 2.1 is established. ■

The following two remarks show that it is essential to assume that  $x$  lies on a fixed curve of type II in Theorem 2.1.

**Remark 4.4** If  $x \in X$  is an isolated fixed point of all iterates  $f^n$ , then the indices  $\nu_x(f^n)$  may depend on  $n \in \mathbb{N}$ . For example, consider a birational map  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  expressed as

$$f(z_1, z_2) = (-2z_1 - z_1^2 - z_2, z_1),$$

in affine coordinates. Then the origin  $(0, 0)$  is an isolated fixed point of all iterates  $f^n$ . Indeed, assume the contrary that an iterate  $f^n$  fixes some curve  $C \subset \mathbb{P}^2$  passing through  $(0, 0)$ . Let  $L \subset \mathbb{P}^2$  be the line at infinity. We observe that  $f$  has a superattracting fixed point  $p^+ \in L$  and  $f$  contracts  $L$  into  $p^+$ . Similarly  $f^{-1}$  has a superattracting fixed point  $p^- \in L$  and  $f^{-1}$  contracts  $L$  into  $p^-$ , where  $p^\pm$  are distinct. The curve  $C$  intersects the line  $L$  in a point, say,  $q \in C \cap L$ . If  $q \neq p^-$  then  $q = f^n(q) = p^+$ , and if  $q \neq p^+$  then  $q = f^{-n}(q) = p^-$ . But both equalities are impossible, because  $p^+$  is an isolated fixed point of  $f^n$  and  $p^-$  is an isolated fixed point of  $f^{-n}$ . Thus  $(0, 0)$  is an isolated fixed point of all iterates  $f^n$ . A little calculation shows that  $\nu_{(0,0)}(f) = 1$  and  $\nu_{(0,0)}(f^2) = 3$  are distinct, though  $\nu_{(0,0)}(f^n)$  are bounded by Theorem 1.1. Note that  $f$  preserves the standard area form  $dz_1 \wedge dz_2$ .

**Remark 4.5** If a point  $x \in X$  is on a fixed curve  $C$  of type I, then the indices  $\nu_x(f^n)$  may depend on  $n \in \mathbb{N}$ . For example, consider a birational map  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  expressed as

$$f(z_1, z_2) = (z_1 + z_1(z_1^2 + z_2), z_2 + z_1^2).$$

in affine coordinates. Then  $C := \{z_1 = 0\}$  is a fixed curve of type I. We can easily check that the index  $\nu_{(0,z_2)}(f^n)$  at  $(0, z_2) \in C$  is positive if and only if  $z_2$  is a root of the equation  $g_n(z_2) := (z_2 + 1)^n - 1 = 0$ . For example,  $\nu_{(0,-2)}(f) = 0$  and  $\nu_{(0,-2)}(f^2) \geq 1$  are distinct.

Moreover, since the equation  $g_n(z_2) = 0$  has  $n$  distinct roots for each  $n \in \mathbb{N}$ , the number of points  $x \in C$  such that  $\nu_x(f) \geq 1$  grows linearly as  $n$  tends to infinity. Thus,

$$\sum_{x \in C} \nu_x(f^n) \rightarrow +\infty \quad (n \rightarrow +\infty).$$

On the other hand, if  $C$  is a fixed curve of type II, Theorem 2.1 and Remark 3.6 imply that

$$\sum_{x \in C} \nu_x(f^n) = \sum_{x \in C} \nu_x(f) < \infty \quad (n \in \mathbb{N}).$$

In order to get the last equality we need the invariance of the indices  $\nu_x(f^n)$  as a function of  $n \in \mathbb{N}$ , whereas the boundedness of  $\nu_x(f^n)$  as in Theorem 1.1 is not enough for this aim.

## 5 Finiteness of Periodic Curves

The aim of this section is to discuss the finiteness of the number of periodic curves of type II and especially to prove Theorem 2.4. Actually we establish more refined results (Theorems 5.3, 5.5 and 5.10), dividing our discussion into three cases according to the values of the first dynamical degree  $\lambda(f)$  of  $f$  and also to the values of the Kodaira dimension  $\text{kod}(X)$  of  $X$ . Theorem 2.4 is then deduced as a corollary of these results. We begin this section with a result on the constraints for the prime periods of two intersecting periodic curves. It plays an important role in the main discussion of this section, while it is also of intrinsic interest in its own light.

**Theorem 5.1** *Let  $f : X \rightarrow X$  be a nontrivial AS birational map and let  $C$  be a periodic curve of type II with prime period  $n$ . If  $C'$  is a periodic curve of prime period  $m$  that intersects  $C$ , then  $m$  is a divisor of  $n$ . If moreover  $C'$  is of type II, then  $m = n$ .*

*Proof.* Assume that  $C$  and  $C'$  intersect in a point  $x \in C \cap C'$ . In view of (16), since the map  $f$  is assumed to be AS, some choice of double signs  $(\varepsilon, \delta) \in \{\pm 1\}^2$  makes  $x \in X_0^\circ(f^{\varepsilon n}) \cap X_0^\circ(f^{\delta m})$ . Among the four cases we only discuss the two cases  $(\varepsilon, \delta) = (+, +)$  and  $(\varepsilon, \delta) = (+, -)$ , as the remaining cases  $(\varepsilon, \delta) = (-, +)$  and  $(\varepsilon, \delta) = (-, -)$  can be treated in similar manners.

First, assume that  $x \in X_0^\circ(f^n) \cap X_0^\circ(f^m)$ , namely, that  $x \in C \setminus I(f^n)$  and  $x \in C' \setminus I(f^m)$ . Then one can think of three endomorphisms:

$$(f^n)_x^*, \quad (f^m)_x^*, \quad ((f^n)_x^*)^m = (f^{nm})_x^* = ((f^m)_x^*)^n : A_x \rightarrow A_x.$$

Since  $C \in X_{II}(f^n)$  passes through  $x$ , any irreducible component of the germ  $C_x$  at  $x$  defines an element of  $\Lambda_{II}((f^n)_x^*)$ , which is therefore nonempty. So Theorem 4.1 with  $\sigma = (f^n)_x^*$  yields

$$\Lambda((f^n)_x^*) = \Lambda(((f^n)_x^*)^m) = \Lambda(((f^m)_x^*)^n) \supset \Lambda((f^m)_x^*).$$

The prime ideal  $\mathfrak{p}$  corresponding to any irreducible component of the germ  $C'_x$  is an element of  $\Lambda((f^m)_x^*)$ . Thus the inclusion relation above yields  $\mathfrak{p} \in \Lambda((f^n)_x^*)$ , which means that  $C' \setminus I(f^n)$  is fixed pointwise by  $f^n$ . Now recall that  $C' \setminus I(f^m)$  is fixed pointwise by  $f^m$ . Write  $n = km + r$  with  $k \in \mathbb{Z}_{\geq 0}$  and  $r \in \{0, 1, \dots, m-1\}$ . Then  $C' \setminus (I(f^n) \cup I(f^m) \cup I(f^r))$  and hence  $C' \setminus I(f^r)$  are fixed pointwise by  $f^r$ . Since  $C'$  is a periodic curve of prime period  $m > r$ , we must have  $r = 0$  and  $n = km$ . Hence  $m$  is a divisor of  $n$ .

Secondly, assume that  $x \in X_0^\circ(f^n) \cap X_0^\circ(f^{-m})$ , namely, that  $x \in C \setminus I(f^n)$  and  $x \in C' \setminus I(f^{-m})$ . Then we have  $x \in X_0^\circ(f^{nm}) \cap X_0^\circ(f^{-mn})$  and hence  $f^{nm}$  defines a local biholomorphism around  $x$ , which induces a ring automorphism  $((f^n)_x^*)^m = (f^{nm})_x^* : A_x \rightarrow A_x$  together with its inverse  $((f^{-m})_x^*)^n = (f^{-nm})_x^* = ((f^{nm})_x^*)^{-1} : A_x \rightarrow A_x$ . By Lemma 3.2 we have  $\Lambda((f^{nm})_x^*) = \Lambda(((f^{nm})_x^*)^{-1})$  and hence  $\Lambda(((f^n)_x^*)^m) = \Lambda(((f^{-m})_x^*)^n) \supset \Lambda((f^{-m})_x^*)$ . On the other hand, since  $C \in X_{II}(f^n)$  passes through  $x$ , the set  $\Lambda_{II}((f^n)_x^*)$  is nonempty. So Theorem 4.1 with  $\sigma = (f^n)_x^*$  implies  $\Lambda((f^n)_x^*) = \Lambda(((f^n)_x^*)^m) \supset \Lambda((f^{-m})_x^*)$ . Now note that  $X_1(f^m) = X_1(f^{-m})$  by (17). Since  $C' \in X_1(f^m) = X_1(f^{-m})$  passes through  $x$ , any irreducible component of the germ  $C'_x$  defines a prime element  $\mathfrak{p} \in \Lambda((f^{-m})_x^*)$ . By the inclusion above we have  $\mathfrak{p} \in \Lambda((f^n)_x^*)$ , which means that  $C' \setminus I(f^n)$  is fixed pointwise by  $f^n$ , while  $C' \setminus I(f^m)$  is fixed pointwise by  $f^m$ . The remaining argument is the same as in the last paragraph. We have  $n = km$  for some  $k \in \mathbb{Z}_{\geq 0}$ .

In any case it is shown that  $m$  is a divisor of  $n$ . If moreover  $C'$  is of type II, then the same reasoning as above with  $C$  replaced by  $C'$  implies that  $n$  is a divisor of  $m$  and hence  $m = n$ . ■

Diller and Favre [5] give a classification of bimeromorphic maps on a compact Kähler surface in terms of their first dynamical degrees (6). We make use of this classification in our discussion.

**Theorem 5.2 ([5])** *Let  $f : X \rightarrow X$  be a bimeromorphic map on a compact Kähler surface  $X$ . Then  $f$  is classified in the following manner up to bimeromorphic conjugacy.*

• *If  $\lambda(f) = 1$ , then exactly one of the following is true:*

- (0)  *$\|(f^n)^*\|$  are bounded and  $f^m$  is an automorphism isotopic to the identity for some  $m \in \mathbb{N}$ ;*
- (1)  *$\|(f^n)^*\|$  grow linearly and  $f$  preserves a unique rational fibration  $\pi : X \rightarrow S$ ;*
- (2)  *$\|(f^n)^*\|$  grow quadratically and  $f$  preserves a unique elliptic fibration  $\pi : X \rightarrow S$ .*

• *If  $\lambda(f) > 1$ , then either*

- (3)  *$X$  is a rational surface with  $f$  an automorphism or merely a bimeromorphic map; or*
- (4)  *$f$  is an automorphism of a K3 surface, an Enriques surface or a complex 2-torus.*

The case  $\lambda(f) = 1$  of low dynamical complexity is more or less easy to handle. So we are mostly concerned with the case  $\lambda(f) > 1$ , which is divided into subcases (3) and (4) according to  $\text{kod}(X) = -\infty$  and  $\text{kod}(X) \geq 0$  respectively. We begin with the last case (4).

**Theorem 5.3** *If  $X$  is a compact Kähler surface of Kodaira dimension  $\text{kod}(X) \geq 0$  and  $f : X \rightarrow X$  is a bimeromorphic map of first dynamical degree  $\lambda(f) > 1$ , then  $f$  has no irreducible periodic curves of nonnegative self-intersection.*

*Proof.* It suffices to show that  $f$  has no irreducible fixed curves of nonnegative self-intersection, because if one wants to consider periodic curves of period  $n$ , then one may replace  $f$  with  $f^n$  upon noting that  $\lambda(f^n) = \lambda(f)^n > 1$ . Now assume the contrary that  $f$  admits an irreducible fixed curve  $C$  of nonnegative self-intersection  $C^2 \geq 0$ . Since  $\text{kod}(X) \geq 0$ , it follows from [2, Chap. VI, (1.1) Theorem] and [5, Proposition 7.5] that there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

such that  $\varphi$  is a proper modification,  $Y$  is the unique minimal model of  $X$  and  $g$  is an automorphism such that  $\lambda := \lambda(g) = \lambda(f) > 1$ . Let  $E(\varphi)$  be the exceptional set of  $\varphi$ . Since any irreducible component of  $E(\varphi)$  has a negative self-intersection, the curve  $C$  is not contained in  $E(\varphi)$ . So  $C' := \varphi(C)$  is an irreducible fixed curve of  $g$ . Since blowing down a curve does not decrease its self-intersection number,  $C'$  has also a nonnegative self-intersection. By [5, Theorem 0.3] there exists a nef class  $\theta \in H^{1,1}(Y)$  such that  $g^*\theta = \lambda\theta$ . Then we have  $g^*C' = C'$  and hence  $\lambda(C', \theta) = (C', \lambda\theta) = (g^*C', g^*\theta) = (C', \theta)$ , where in the last equality we use the fact that an automorphism preserves the intersection form. But, since  $\lambda > 1$ , we have  $(C', \theta) = 0$  together with  $(C')^2 \geq 0$  and  $\theta^2 \geq 0$ . Then the Hodge index theorem and [2, Chap. IV, (7.2) Corollary] imply that  $(C')^2 = \theta^2 = (C', \theta) = 0$  and there exists a positive constant  $a > 0$  such that  $C' = a\theta$  in  $H^{1,1}(Y)$ . Applying  $g^*$  to this equality, we have  $C' = a\lambda\theta$ . But, since  $\lambda > 1$ , we have  $C' = \theta = 0$  in  $H^{1,1}(Y)$ . This is a contradiction. ■

**Remark 5.4** If  $\text{kod}(X) = -\infty$ , Theorem 5.3 is not true in general. For example, consider the birational map  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  discussed in Remark 4.5. We can see that  $f^{-m}I(f) = \{[0 : 0 : 1]\}$  and  $f^n I(f^{-1}) = \{[0 : 1 : 0], [1 : 0 : -1]\}$  for every  $m, n \geq 0$  and so  $f$  is AS. Since  $\deg f = 3$ , we have  $\lambda(f) = 3 > 1$ . The map  $f$  has the line  $C := \{z_1 = 0\}$  as a fixed curve of self-intersection  $C^2 = 1$ . This gives a counterexample to Theorem 5.3 when  $X$  is a rational surface, or more specifically when  $X = \mathbb{P}^2$ . We can also construct an AS birational map of a rational surface having a fixed curve of zero self-intersection by blowing up a suitable point of  $C$ , say,  $[1 : 0 : 0] \in C$ , and lifting the map  $f$  to the surface upstairs.

We proceed to the case where  $\lambda(f) > 1$  and  $\text{kod}(X) = -\infty$ , namely, the case (3).

**Theorem 5.5** *If  $X$  is a smooth rational surface and  $f : X \rightarrow X$  is an AS birational map of first dynamical degree  $\lambda(f) > 1$ , then all the irreducible periodic curves of type II of  $f$  with zero self-intersection have one and the same prime period.*

The proof is divided into several steps and begins with some generality on fibrations.

**Lemma 5.6** *Let  $\pi : X \rightarrow S$  be a fibration with connected fibers of a smooth surface  $X$  and let  $C \subset X$  be a curve with zero self-intersection such that  $\pi(C) = \{t\}$ . Then the following hold:*

- (1)  $X_t := \pi^{-1}(t) = sC$  for some  $s \in \mathbb{N}$ .
- (2) *If a connected curve  $C'$  is disjoint from  $C$ , then  $C'$  is contained in a fiber of  $\pi$ .*

*Proof.* Assertion (1): Write  $X_t = sC + D$ , where  $D$  is an effective divisor not containing  $C$ . Assume that  $D$  is nonempty. Since each fiber of  $\pi$  is connected,  $C$  and  $D$  intersect so that  $C \cdot D > 0$ . Then  $(mC + D)^2 = 2mC \cdot D + D^2 > 0$  for a sufficiently large integer  $m \in \mathbb{N}$ . This contradicts the fact that any divisor supported on a fiber has a nonpositive self-intersection (see [2, Chap. III, (8.2) Lemma]). Hence  $D$  must be empty and assertion (1) is proved.

Assertion (2): Since  $C'$  is connected, its image  $\pi(C')$  is a connected algebraic subset of  $S$ , which must be a single point of  $S$  or the entire curve  $S$ . On the other hand, since  $C'$  is disjoint from  $C$ , assertion (1) implies that  $C'$  is also disjoint from  $X_t$  and hence  $\pi(C') \subset S \setminus \{t\}$ . Therefore  $\pi(C')$  must be a single point, so that  $C'$  is contained in a fiber of  $\pi$ . ■

To prove Theorem 5.5 by contradiction, we assume the contrary and proceed as follows.

**Lemma 5.7** *Let  $X$  be a smooth rational surface and  $f : X \rightarrow X$  an AS birational map with  $\lambda(f) > 1$ . Assume that  $f$  admits two irreducible periodic curves  $C_1$  and  $C_2$  of type II with zero self-intersection and with distinct prime periods  $n_1$  and  $n_2$  respectively. Then there exists a fibration  $\pi : X \rightarrow S$  with connected fibers such that  $C_1$  and  $C_2$  are fibers of  $\pi$ . Moreover, if a connected curve  $C \subset X$  is disjoint from  $C_1$  or  $C_2$ , then  $C$  is contained in a fiber of  $\pi$ .*

*Proof.* Since  $C_1$  and  $C_2$  are periodic curves of type II with distinct prime periods, Theorem 5.1 implies that  $C_1 \cap C_2 = \emptyset$ . This together with the assumption of zero self-intersection yields  $C_1^2 = C_2^2 = (C_1, C_2) = 0$ . Then Hodge index theorem tells us that  $C_1$  and  $C_2$  are linearly dependent in  $\text{NS}(X) \otimes \mathbb{R}$ , where  $\text{NS}(X)$  is the Néron-Severi group of  $X$ . Moreover, since  $C_1$  and  $C_2$  are positive and  $\text{NS}(X)$  is defined over  $\mathbb{Z}$ , there exist  $a_1, a_2 \in \mathbb{N}$  such that  $a_1 C_1 = a_2 C_2$  in  $\text{NS}(X)$ . Since  $X$  is a rational surface, we have  $H^1(X, \mathcal{O}_X) = 0$ , so that the first Chern class map  $c_1 : \text{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \rightarrow \text{NS}(X)$  is injective. Thus  $a_1 C_1 = a_2 C_2$  in  $\text{Pic}(X)$ , namely, the divisors  $a_1 C_1$  and  $a_2 C_2$  are linearly equivalent. Then there exists a surjective holomorphic map  $\tilde{\pi} : X \rightarrow \mathbb{P}^1$  such that  $\tilde{\pi}^{-1}(0) = a_1 C_1$  and  $\tilde{\pi}^{-1}(\infty) = a_2 C_2$ . Using the Stein factorization (see [2, Chap. I, (8.1) Theorem]), we obtain a fibration  $\pi : X \rightarrow S$  over some curve  $S$  with connected fibers and a finite morphism  $\phi : S \rightarrow \mathbb{P}^1$  such that  $\tilde{\pi} = \phi \circ \pi : X \rightarrow \mathbb{P}^1$ . Since  $C_i$  is a connected fiber of  $\tilde{\pi}$ , it is also a fiber of the fibration  $\pi$ . Finally, if a connected curve  $C$  is disjoint from  $C_i$  for at least one  $i \in \{1, 2\}$ , then assertion (2) of Lemma 5.6 implies that  $C$  is contained in a fiber of the fibration  $\pi$ . ■

For any curve  $C$  on a surface  $X$ , one can define the pushforward  $f_* C$  of  $C$  by pulling back the local defining function of  $C$  by  $f^{-1}$ . In general,  $f_* C - fC$  is a nonnegative linear combination of irreducible components of  $E(f^{-1})$ . In this situation, Diller and Favre [5, Corollary 3.4] obtain a useful formula for the intersection number of the pushforwards of two curves:

$$(f_* C, f_* C') = (C, C') + Q(C, C'), \quad (30)$$

where  $Q(C, C')$  is a nonnegative Hermitian form expressed as

$$Q(C, C') = \sum_{\substack{V \subset E(f): \\ \text{irreducible}}} k(V) \cdot (C, V) \cdot (C', V),$$

with some positive integer  $k(V) \in \mathbb{N}$  for each irreducible component  $V$  of  $E(f)$ . Note that  $Q(C, C) = 0$  if and only if  $(C, V) = 0$  for any irreducible component  $V$  of  $E(f)$ .

**Lemma 5.8** *Under the assumptions of Lemma 5.7, for any  $i \geq 0$ , the  $i$ -th pushforward  $f_*^i C_1$  has zero self-intersection;  $(f_*^i C_1, V) = 0$  for any irreducible component  $V$  of  $E(f)$ ; and  $f_*^i C_1$  is equal to a fiber of the fibration  $\pi : X \rightarrow S$  constructed in Lemma 5.7.*

*Proof.* For each  $i \geq 0$  we put  $f_*^i C_1 = f^i C_1 + E_i$  with some effective divisor  $E_i$ . Since  $f^i C_1$  is a periodic curve of prime period  $n_1$ , Theorem 5.1 implies that  $f^i C_1$  is disjoint from  $C_2$ . We see that  $E_i$  is also disjoint from  $C_2$ . Indeed, if  $E_i$  intersects  $C_2$ , then  $f^{-i} C_2$  must intersect  $C_1$ , but this contradicts the fact that  $f^{-i} C_2$  is a periodic curve of prime period  $n_2$ . Thus  $f_*^i C_1 = f^i C_1 + E_i$  is disjoint from  $C_2$ . Since  $f_*^i C_1$  is connected, Lemma 5.7 implies that  $f_*^i C_1$  is contained in a fiber of the fibration  $\pi$ . In particular,  $f_*^{n_1} C_1 = f^{n_1} C_1 + E_{n_1} = C_1 + E_{n_1}$  is contained in a fiber of  $\pi$ . Because the fiber containing  $C_1$  is  $C_1$  itself by Lemma 5.7, we have



$E_{n_1} = \emptyset$  and thus  $f_*^{n_1}(C_1) = C_1$ . A repeated use of formula (30) yields

$$\begin{aligned} C_1^2 &= (f_*^{n_1} C_1)^2 = (f_*^{n_1-1} C_1)^2 + Q(f_*^{n_1-1} C_1, f_*^{n_1-1} C_1) = \dots \\ &= (f_*^j C_1)^2 + \sum_{i=j}^{n_1-1} Q(f_*^i C_1, f_*^i C_1) = \dots \\ &= C_1^2 + \sum_{i=0}^{n_1-1} Q(f_*^i C_1, f_*^i C_1). \end{aligned}$$

Since  $Q$  is a nonnegative Hermitian form, this formula shows that  $Q(f_*^i C_1, f_*^i C_1) = 0$  for any  $0 \leq i \leq n_1 - 1$  and thus  $(f_*^i C_1)^2 = 0$  and  $(f_*^i C_1, V) = 0$  for any  $0 \leq i \leq n_1 - 1$  and any irreducible component  $V$  of  $E(f)$ . These are true for any  $i \geq 0$  since  $f_*^{n_1} C_1 = C_1$ . By assertion (1) of Lemma 5.6 we can conclude that  $f_*^i C_1$  is a fiber of  $\pi$ . The proof is complete. ■

**Lemma 5.9** *Under the assumptions of Lemma 5.7, the map  $f : X \rightarrow X$  preserves the fibration  $\pi : X \rightarrow S$  constructed in Lemma 5.7.*

*Proof.* Fix a point  $t \in S$  and consider the fiber  $X_t := \pi^{-1}(t)$  over  $t$ . We show that  $f_* X_t$  is contained in a fiber of the fibration  $\pi$ . This is true for  $X_t = C_1$  by Lemma 5.7, so that we may assume that  $X_t \cap C_1 = \emptyset$ . By formula (30) and Lemma 5.8, we have

$$(f_* X_t, f_* C_1) = (X_t, C_1) + \sum_{\substack{V \subset E(f): \\ \text{irreducible}}} k(V) \cdot (X_t, V) \cdot (C_1, V) = (X_t, C_1) = 0.$$

This means that  $f_* X_t$  is disjoint from  $f_* C_1$ . Since  $f_* C_1$  is equal to a fiber of  $\pi$  by Lemma 5.8,  $f_* X_t$  is contained in a fiber of  $\pi$  by Lemma 5.6. ■

*Proof of Theorem 5.5.* Assume the contrary that the theorem does not hold. Then the assumptions of Lemma 5.7 are satisfied and hence  $f$  preserves a fibration  $\pi : X \rightarrow S$  by Lemma 5.9. But this is absurd, because a bimeromorphic map preserving a fibration has first dynamical degree  $\lambda(f) = 1$  (see [4, Corollary 1.3]). Thus the theorem is established. ■

**Theorem 5.10** *Let  $X$  be a compact Kähler surface and  $f : X \rightarrow X$  a bimeromorphic map with  $\lambda(f) = 1$  such that  $f^n$  is not isotopic to the identity for any  $n \in \mathbb{N}$ . Let  $\pi : X \rightarrow S$  be the unique rational or elliptic fibration preserved by  $f$  in Theorem 5.2. If  $f$  admits two irreducible periodic curves  $C_1$  and  $C_2$  of type II with zero self-intersection and with distinct prime periods  $n_1$  and  $n_2$  respectively, then any irreducible periodic curve of type II with an arbitrary prime period and any irreducible periodic curve of type I whose prime period is not a common divisor of  $n_1$  and  $n_2$  are contained in fibers of the fibration  $\pi$ .*

*Proof.* First notice that  $C_1$  and  $C_2$  are disjoint by Theorem 5.1, since  $n_1$  and  $n_2$  are distinct. Let  $C$  be a periodic curve of type II with an arbitrary prime period  $n$  or a periodic curve of type I whose prime period  $n$  is not a common divisor of  $n_1$  and  $n_2$ . We claim that  $C$  is disjoint from either  $C_1$  or  $C_2$ . Indeed, if  $C$  is of type II, then its prime period  $n$  is different from  $n_i$  for at least one  $i \in \{1, 2\}$ , and hence Theorem 5.1 shows that  $C$  is disjoint from  $C_i$ . On the other hand, if  $C$  is of type I and meets both  $C_1$  and  $C_2$ , then  $n$  must divide both  $n_1$  and  $n_2$ , which contradicts the assumption by Theorem 5.1. Hence the claim is verified.

We establish the theorem by a case-by-case check. First, when  $\pi : X \rightarrow S$  is an elliptic fibration, it follows from [6, Theorem 3.4] that  $C$  is contained in a fiber of the fibration  $\pi$ . Next we consider the case where  $\pi : X \rightarrow S$  is a rational fibration. Assume that  $\pi(C_i) = S$  and  $C_j$  is contained in a fiber of  $\pi$  for some  $\{i, j\} = \{1, 2\}$ , then  $C_j$  is equal to a fiber of  $\pi$  by assertion (1) of Lemma 5.6 and thus  $C_i$  must intersect  $C_j$ , but this contradicts the fact that  $C_1 \cap C_2 = \emptyset$ . Now assume that  $\pi(C_1) = \pi(C_2) = S$ . Then any fiber  $X_t := \pi^{-1}(t)$  of  $\pi$  meets both  $C_1$  and  $C_2$ . If we take  $t \in S$  to be sufficiently generic, then  $X_t \cong \mathbb{P}^1$  and an intersection point  $p_i \in X_t \cap C_i$  becomes a periodic point of  $f$  with prime period  $n_i$  for each  $i \in \{1, 2\}$ . Since  $f$  preserves the fibration  $\pi$  and  $f^{n_i}(p_i) = p_i$ , we have two automorphisms  $f^{n_i}|_{X_t} : X_t \rightarrow X_t$  ( $i = 1, 2$ ). If  $d$  denotes the greatest common divisor of  $n_1$  and  $n_2$ , then  $f^d|_{X_t}$  becomes an automorphism of  $X_t$  having  $p_1$  and  $p_2$  as periodic points of prime periods  $n_1/d$  and  $n_2/d$  respectively. Hence  $f^d|_{X_t}$  is a linear fractional transformation with two periodic points of distinct prime periods, but this is impossible. Thus for each  $i = 1, 2$ , the curve  $C_i$  must be contained in a fiber of  $\pi$  and in fact equal to that fiber by assertion (1) of Lemma 5.6. Since  $C$  is disjoint from either  $C_1$  or  $C_2$ , assertion (2) of Lemma 5.6 implies that  $C$  is contained in a fiber of  $\pi$ . ■

Finally, in order to prove Theorem 2.4, we need the following lemma.

**Lemma 5.11** *Let  $k \in \mathbb{N}$ . Given  $\rho(X) + k$  irreducible periodic curves of type II with mutually distinct prime periods, then at least  $k$  of them have zero self-intersection.*

*Proof.* Let  $C_1, \dots, C_{\rho(X)+k}$  be the periodic curves of type II with mutually distinct prime periods. They are mutually disjoint by Theorem 5.1. If the contrary to the lemma holds, then we may assume that  $C_i^2$  is nonzero for every  $i \in \{1, 2, \dots, \rho(X) + 1\}$  after rearranging the suffixes if necessary. Since  $\rho(X) = \dim_{\mathbb{R}} \text{NS}(X) \otimes \mathbb{R}$ , there is a nontrivial linear relation  $r_1 C_1 + \dots + r_{\rho(X)+1} C_{\rho(X)+1} = 0$  in  $\text{NS}(X) \otimes \mathbb{R}$ . Since  $C_i$  and  $C_j$  are disjoint for every distinct  $i$  and  $j$ , the linear relation yields

$$0 = \sum_{j=1}^{\rho(X)+1} r_j (C_i, C_j) = r_i C_i^2 \quad (i = 1, \dots, \rho(X) + 1),$$

which means that  $r_1 = \dots = r_{\rho(X)+1} = 0$ . This contradicts the fact that the linear relation is nontrivial. Thus at least  $k$  of  $C_1, \dots, C_{\rho(X)+k}$  have zero self-intersection. ■

*Proof of Theorem 2.4.* Assertion (1). First we consider the case  $\text{kod}(X) \geq 0$ . Assume that  $f$  admits  $\rho(X) + 1$  irreducible periodic curves of type II with mutually distinct prime periods. Then Lemma 5.11 with  $k = 1$  implies that at least one of them has zero self-intersection. But this is impossible by Theorem 5.3. Thus there are at most  $\rho(X)$  irreducible periodic curves of type II with mutually distinct prime periods. This proves the item (2) of Remark 2.5. Next we consider the case  $\text{kod}(X) = -\infty$ , namely, the case where  $X$  is rational. Assume that  $f$  admits  $\rho(X) + 2$  irreducible periodic curves of type II with mutually distinct prime periods. Then Lemma 5.11 with  $k = 2$  implies that at least two of them have zero self-intersection. But this is impossible by Theorem 5.5. Thus there are at most  $\rho(X) + 1$  irreducible periodic curves of type II with mutually distinct prime periods. Therefore assertion (1) of the theorem is proved.

Assertion (2). Assume that  $f$  has more than  $\rho(X) + 1$  irreducible periodic curves of type II with mutually distinct prime periods. Again by Lemma 5.11 with  $k = 2$ , at least two of them have zero self-intersection. Then Theorem 5.10 implies that any irreducible periodic curve of type II is contained in a fiber of the fibration  $\pi$ . ■

## 6 Area-Preserving Maps

The aim of this section is to discuss the absence of periodic curves of type I for an area-preserving map and to prove Theorems 2.4. Given a fixed curve  $C \in X_1(f)$ , we take a smooth point  $x$  of  $C$  and identify  $A_x$  with  $\mathbb{C}[[z_1, z_2]]$  in such a manner that  $C$  has the local defining equation  $z_1 = 0$  near  $x$ . Then the induced endomorphism  $f_x^* : A_x \rightarrow A_x$  can be expressed as

$$\begin{cases} f_x^*(z_1) &= z_1 + z_1^k \cdot f_1, \\ f_x^*(z_2) &= z_2 + z_1^l \cdot f_2, \end{cases} \quad (31)$$

for some  $k, l \in \mathbb{N} \cup \{\infty\}$  and some  $f_i \in A_x$  such that  $f_i(0, z_2)$  is a nonzero element of  $\mathbb{C}[[z_2]]$ . Here we put  $z_1^\infty := 0$  by convention and we remark that at least one of  $k$  and  $l$  is finite.

**Lemma 6.1** *For  $C \in X_1(f)$ , we have  $\nu_C(f) = \min\{k, l\}$  and  $C \in X_{II}(f)$  if and only if  $k > l$ .*

*Proof.* In (31) we put  $f_i = u \cdot v_i$ , where  $v_1$  and  $v_2$  are relatively prime. If  $k > l$ , then comparing (8) with (31) we have  $g = z_1^l \cdot u$ ,  $h_1 = z_1^{k-l} \cdot v_1$  and  $h_2 = v_2$ . Hence (10) and (11) yield

$$\begin{aligned} \nu_C(f) &= \max\{m \in \mathbb{N} \mid (g) \subset (z_1)^m\} = l, \\ \varpi_{f_x^*} &= v_2 \cdot dz_1 - z_1^{k-l} \cdot v_1 \cdot dz_2 \in \hat{\Omega}_{A_x/\mathbb{C}}^1, \end{aligned}$$

which shows that  $\tau_{(z_1)}(\varpi_{f_x^*}) = 0$  and hence  $C$  is of type II relative to  $f$ . On the other hand, if  $k \leq l$ , then we have  $g = z_1^k \cdot u$ ,  $h_1 = v_1$  and  $h_2 = z_1^{l-k} \cdot v_2$ . Hence (10) and (11) yield

$$\begin{aligned} \nu_C(f) &= \max\{m \in \mathbb{N} \mid (g) \subset (z_1)^m\} = k, \\ \varpi_{f_x^*} &= z_1^{l-k} \cdot v_2 \cdot dz_1 - v_1 \cdot dz_2 \in \hat{\Omega}_{A_x/\mathbb{C}}^1, \end{aligned}$$

which shows that  $\tau_{(z_1)}(\varpi_{f_x^*}) = -v_1 \cdot dz_2 \neq 0$  and hence  $C$  is of type I relative to  $f$ . ■

Using this lemma we complete the proof of Theorem 2.6

*Proof of Theorem 2.6.* Assume that the 2-form  $\omega$  has no pole of order  $\nu_C(f)$  along  $C$ . In view of Lemma 6.1 the theorem is proved if  $k > l$  is shown in (31). Around  $x$  we express  $\omega$  as

$$\omega = \alpha \cdot dz_1 \wedge dz_2 \quad \text{with} \quad \alpha = \sum_{n=s}^{\infty} \alpha_n(z_2) z_1^n \quad (32)$$

where  $s \in \mathbb{Z}$  and  $\alpha_n(z_2) \in \mathbb{C}((z_2))$  with  $\alpha_s(z_2)$  not identically zero. Assume the contrary that  $k \leq l$ . Then we have  $\nu_C(f) = k$  by Lemma 6.1 and hence  $k \neq -s$ , since  $\omega$  has a pole of order  $-s$  along  $C$  (a pole of negative order is a zero). In order to consider the area-preserving property  $f_x^* \omega = \omega$ , we calculate  $f_x^* \omega := \alpha(f_x^*(z_1), f_x^*(z_2)) \cdot d(f_x^*(z_1)) \wedge d(f_x^*(z_2))$ . Considering the Laurent expansion of  $\alpha(f_x^*(z_1), f_x^*(z_2))$  in  $z_1$ , we have

$$\begin{aligned} \alpha(f_x^*(z_1), f_x^*(z_2)) &= \sum_{n=s}^{\infty} \alpha_n(z_2 + z_1^l \cdot f_2) \cdot (z_1 + z_1^k \cdot f_1)^n \\ &= \sum_{i,j=0}^{\infty} \sum_{n=s}^{\infty} \binom{n}{i} \frac{\alpha_n^{(j)}(z_2)}{j!} \cdot f_1^i \cdot f_2^j \cdot z_1^{n+(k-1)i+l j} \\ &= \sum_{n=s}^{s+k-1} \alpha_n(z_2) \cdot z_1^n + s \cdot \alpha_s(z_2) \cdot f_1 \cdot z_1^{s+k-1} + O(z_1^{s+k}), \end{aligned}$$

where on the righthand side the first terms come from the indices  $(i, j) = (0, 0)$ ,  $s \leq n \leq s+k-1$ , the second term from  $(i, j) = (1, 0)$ ,  $n = s$ , and the  $O(z_1^{s+k})$ -term from the remaining indices, since we are assuming that  $k-1 < l$ . Similarly it follows from  $1 \leq k \leq l$  that

$$\begin{aligned} d(f_x^*(z_1)) \wedge d(f_x^*(z_2)) &= \{ (1 + k \cdot z_1^{k-1} \cdot f_1 + z_1^k \cdot f_{1z_1}) \cdot (1 + z_1^l \cdot f_{2z_2}) \\ &\quad - z_1^k \cdot f_{1z_2} \cdot (l \cdot z_1^{l-1} \cdot f_2 + z_1^l \cdot f_{2z_2}) \} \cdot dz_1 \wedge dz_2 \\ &= \{ 1 + k z_1^{k-1} \cdot f_1 + O(z_1^k) \} \cdot dz_1 \wedge dz_2, \end{aligned}$$

where  $f_{iz_j}$  is the partial derivative of  $f_i$  with respect to  $z_j$ . These calculations and (32) yield

$$\begin{aligned} f_x^* \omega &= \left\{ \sum_{n=s}^{s+k-1} \alpha_n(z_2) \cdot z_1^n + (s+k) \cdot \alpha_s(z_2) \cdot f_1 \cdot z_1^{s+k-1} + O(z_1^{s+k}) \right\} \cdot dz_1 \wedge dz_2, \\ \omega &= \left\{ \sum_{n=s}^{s+k-1} \alpha_n(z_2) \cdot z_1^n + O(z_1^{s+k}) \right\} \cdot dz_1 \wedge dz_2. \end{aligned}$$

Comparing the coefficients of  $z_1^{s+k-1} \cdot dz_1 \wedge dz_2$  in the area-preserving condition  $f_x^* \omega = \omega$ , we have  $(s+k) \cdot \alpha_s(z_2) \cdot f_1(0, z_2) = 0$  in  $\mathbb{C}((z_2))$ , but this contradicts the fact that  $s+k \neq 0$  in  $\mathbb{Z}$  and  $\alpha_s(z_2) \neq 0$ ,  $f_1(0, z_2) \neq 0$  in  $\mathbb{C}((z_2))$ . Therefore actually we have  $k > l$  and hence  $C$  is of type II relative to  $f$  by Lemma 6.1. The proof is complete. ■

**Remark 6.2** If  $C$  is a fixed curve along which  $\omega$  has a pole of order  $\nu_C(f)$ , then  $C$  may be of type I. For example, consider the birational map  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  mentioned in Remark 4.5. Notice that  $f$  preserves the meromorphic 2-form  $\omega := z_1^{-1} dz_1 \wedge dz_2$ . The curve  $C := \{z_1 = 0\}$  is a fixed curve of type I with index  $\nu_C(f) = 1$ , along which  $\omega$  has a pole of order  $\nu_C(f) = 1$ .

In Section 1 a discussion is made rather roughly as to the inapplicability of some generalized fixed formulas other than Saito's formula (5). We take this occasion to restate it more precisely.

**Remark 6.3** The generalized fixed point formulas cited in Section 1 other than formula (5) are valid when  $X$  is a complex manifold and  $f : X \rightarrow X$  is a holomorphic map such that each connected component  $Y$  of the fixed point set is a non-degenerate submanifold, that is, all the eigenvalues of the normal map  $d^N f : NY \rightarrow NY$  are different from 1, where  $d^N f$  is the map induced from the tangent map  $df : TX \rightarrow TX$  to the normal bundle  $NY := T_Y X / T_Y$ . Under this non-degeneracy condition, if  $X$  is a surface, then Lemma 6.1 readily shows that any fixed curve of  $f$  is of type I. On the other hand, if  $f$  is an area-preserving surface map, then any fixed curve  $C$  must be degenerate, because the normal map  $d^C f$  becomes identity. In fact we know from Theorem 2.6 that  $C$  is of type II. Thus the area-preserving surface maps are completely outside the reach of the usual generalized fixed point formulas other than formula (5).

## 7 Isolated Periodic Points

In this section, based on the fundamental results in Theorems 1.2, 2.1, 2.4, and 5.1, we are concerned with the isolated periodic points of a birational surface map  $f$ . In view of Theorem

2.6 it is reasonable to assume that  $f$  has no periodic curves of type I. Under this assumption we establish a certain periodic point formula (Theorem 7.2), a Shub-Sullivan type result (Lemma 7.6) and a refined version of Theorem 2.8 (Theorem 7.9).

To this end we need to introduce some terminology and notation. Let  $X_0^c(f)$  be the set of all non-isolated fixed points of  $f$ , namely, the set of all points that lie on some fixed curve of  $f$ , and let  $X_0^i(f)$  be the set of all isolated fixed points of  $f$ , that is, the complement of  $X_0^c(f)$  in  $X_0(f)$ . Then  $\text{Per}_n^i(f) := X_0^i(f^n)$  stands for the set of all isolated periodic points of  $f$  with (not necessarily prime) period  $n$  and its cardinality counted with multiplicity is defined by

$$\#\text{Per}_n^i(f) := \sum_{x \in \text{Per}_n^i(f)} \nu_x(f^n). \quad (33)$$

Here a remark on notation:  $\#$  is used to denote the cardinality counted with multiplicity or, in other words, the weighted cardinality, while  $\text{Card}$  is reserved for the cardinality without multiplicity taken into account. We denote by  $P(f)$  the set of all positive integers that arise as the prime period of some irreducible periodic curve of  $f$ . For each  $n \in \mathbb{N}$ , let  $P_n(f)$  be the set of all elements  $k \in P(f)$  that divides  $n$ . Note that  $P_n(f)$  is a finite set for every  $n \in \mathbb{N}$ , while  $P(f)$  may or may not be finite.

**Remark 7.1** The map  $f$  admits infinitely many irreducible periodic curves if and only if the set  $P(f)$  is infinite, because the number of irreducible periodic curves of any given prime period is finite, provided that  $f$  is nontrivial. (Recall that  $f$  is always assumed to be nontrivial.)

Moreover, for each  $k \in P(f)$  we denote by  $\text{PC}_k(f)$  the set of all irreducible periodic curves of  $f$  with prime period  $k$ . There is then a direct sum decomposition:

$$X_1(f^n) = \coprod_{k \in P_n(f)} \text{PC}_k(f). \quad (34)$$

Given any  $k \in P(f)$ , let  $C_k(f)$  be the (possibly reducible) curve in  $X$  defined to be the union of all curves in  $\text{PC}_k(f)$ . Then there exists a decomposition:

$$X_0(f^n) = \text{Per}_n^i(f) \amalg \bigcup_{k \in P_n(f)} C_k(f). \quad (35)$$

Finally, for each  $k \in P(f)$ , let  $\xi_k(f)$  be the number defined by

$$\xi_k(f) := \sum_{x \in C_k(f)} \nu_x(f^k) + \sum_{C \in \text{PC}_k(f)} \tau_C \cdot \nu_C(f^k). \quad (36)$$

With these preliminaries, under the absence of periodic curves of type I, Saito's fixed point formula (5) is applied to the iterates  $f^n$  to yield the following periodic point formula.

**Theorem 7.2** *Let  $f : X \rightarrow X$  be an AS birational map of a smooth projective surface  $X$ . If  $f$  has no periodic curve of type I, then we have for any  $n \in \mathbb{N}$ ,*

$$L(f^n) = \#\text{Per}_n^i(f) + \sum_{k \in P_n(f)} \xi_k(f). \quad (37)$$

*Proof.* By assumption, the map  $f^n$  has no fixed curves of type I, that is,  $X_1(f^n) = X_{II}(f^n)$  for any  $n \in \mathbb{N}$ . Then, by Theorem 5.1, if  $k, l \in P(f)$  are distinct then  $C_k(f)$  and  $C_l(f)$  are disjoint. Thus (35) becomes the direct sum decomposition:

$$X_0(f^n) = \text{Per}_n^i(f) \amalg \coprod_{k \in P_n(f)} C_k(f). \quad (38)$$

Since  $f$  is assumed to be AS, Remark 3.8 implies that the fixed point formula (5) can be applied to all iterates  $f^n$  ( $n \in \mathbb{N}$ ). In view of the direct sum decompositions (34) and (38) and the equality  $X_1(f^n) = X_{II}(f^n)$ , the formula (5) is rewritten as

$$\begin{aligned} L(f^n) &= \sum_{x \in X_0(f^n)} \nu_x(f^n) + \sum_{C \in X_1(f^n)} \tau_C \cdot \nu_C(f^n) \\ &= \sum_{x \in \text{Per}_n^i(f)} \nu_x(f^n) + \sum_{k \in P_n(f)} \sum_{x \in C_k(f)} \nu_x(f^n) + \sum_{k \in P_n(f)} \sum_{C \in \text{PC}_k(f)} \tau_C \cdot \nu_C(f^n) \\ &= \#\text{Per}_n^i(f) + \sum_{k \in P_n(f)} \left\{ \sum_{x \in C_k(f)} \nu_x((f^k)^{n/k}) + \sum_{C \in \text{PC}_k(f)} \tau_C \cdot \nu_C((f^k)^{n/k}) \right\}. \end{aligned}$$

Here we note that  $n/k \in \mathbb{N}$  for any  $k \in P_n(f)$ , any  $x \in C_k(f)$  passes through a fixed curve of type II of  $f^k$  and any  $C \in \text{PC}_k(f)$  is a fixed curve of type II of  $f^k$ . Thus Theorem 2.1 implies that  $\nu_x((f^k)^{n/k}) = \nu_x(f^k)$  and  $\nu_C((f^k)^{n/k}) = \nu_C(f^k)$ . Hence we have

$$\begin{aligned} L(f^n) &= \#\text{Per}_n^i(f) + \sum_{k \in P_n(f)} \left\{ \sum_{x \in C_k(f)} \nu_x(f^k) + \sum_{C \in \text{PC}_k(f)} \tau_C \cdot \nu_C(f^k) \right\} \\ &= \#\text{Per}_n^i(f) + \sum_{k \in P_n(f)} \xi_k(f), \end{aligned}$$

where (36) is used in the last line. This proves the theorem. ■

We need a bit more terminology. Let  $\text{Per}^i(f)$  denote the set of all isolated periodic points of  $f$ , that is, the union of  $\text{Per}_n^i(f)$  over all  $n \in \mathbb{N}$ . We make the following definition.

**Definition 7.3** An isolated periodic point  $x \in \text{Per}^i(f)$  is said to be *absolutely isolated* if  $x$  is an isolated fixed point of  $f^n$  for any period  $n$  of  $x$ . Otherwise  $x$  is said to be *conditionally isolated*. Denote by  $\text{Per}^{ai}(f)$  and  $\text{Per}^{ci}(f)$  the set of all absolutely isolated periodic points and the set of all conditionally isolated periodic points of  $f$  respectively. Then we have

$$\text{Per}^i(f) = \text{Per}^{ai}(f) \amalg \text{Per}^{ci}(f).$$

For any  $x \in \text{Per}^{ci}(f)$  there exists a period  $n \in \mathbb{N}$  of  $x$  relative to  $f$  such that  $x$  is a non-isolated fixed point of  $f^n$ . The smallest such  $n$  is called the *secondary period* of  $x$  relative to  $f$ . Put

$$\begin{aligned} \text{Per}_n^{ai}(f) &:= \text{Per}^{ai}(f) \cap \text{Per}_n^i(f), & \#\text{Per}_n^{ai}(f) &:= \sum_{x \in \text{Per}_n^{ai}(f)} \nu_x(f^n), \\ \text{Per}_n^{ci}(f) &:= \text{Per}^{ci}(f) \cap \text{Per}_n^i(f), & \#\text{Per}_n^{ci}(f) &:= \sum_{x \in \text{Per}_n^{ci}(f)} \nu_x(f^n). \end{aligned}$$

**Remark 7.4** The secondary period  $m$  of  $x \in \text{Per}^{ci}(f)$  is a strictly greater multiple of its prime period  $n$ . Indeed, if  $m = n$  then  $x$  would be a non-isolated fixed point of  $f^{ln}$  for any  $l \in \mathbb{N}$ , contradicting the assumption that  $x$  is an isolated periodic point. Write  $m = kn$  with  $k \in \mathbb{N}_{\geq 2}$ . Then  $x$  is an isolated fixed point of  $f^{ln}$  if and only if  $l$  is not divisible by  $k$ .

Shub and Sullivan [15] state their result just as in Theorem 1.1, but a careful check of their proof shows that their result is valid in the following more general form.

**Theorem 7.5** *Let  $f : X \rightarrow X$  be a  $C^1$ -map of a smooth manifold  $X$  and  $x \in X$  an isolated fixed point of  $f$ . Let  $N_x(f)$  be the set of all  $n \in \mathbb{N}$  such that  $x$  is an isolated fixed point of  $f^n$ . Then the indices  $\nu_x(f^n)$  are bounded as a function of  $n \in N_x(f)$ .*

A combination of Theorems 2.1 and 7.5 leads to the following lemma.

**Lemma 7.6** *Assume that  $f$  has no periodic curves of type I. Given any  $x \in \text{Per}^i(f)$ , let  $n$  be the prime period of  $x$  relative to  $f$ . Then the indices  $\nu_x(f^{ln})$  are bounded as a function of  $l \in \mathbb{N}$ .*

*Proof.* First, if  $x \in \text{Per}^{ai}(f)$ , then the lemma is proved by applying Theorem 1.1 to the map  $f^n$ . Next consider the case  $x \in \text{Per}^{ci}(f)$ . Let  $m$  be the secondary period of  $x$  relative to  $f$  and put  $m = kn$  with  $k \in \mathbb{N}_{\geq 2}$ . Note that  $N_x(f^n) = \{l \in \mathbb{N} \mid l \text{ is not divisible by } k\}$  by Remark 7.4. Then Theorem 7.5 applied to  $f^n$  implies that  $\nu_x(f^{ln})$  is bounded for  $l \in N_x(f^n)$ . For  $l$  divisible by  $k$ , we can use Theorem 2.1 since  $x$  is a fixed point of  $f^m = f^{kn}$  through which a fixed curve of type II of  $f^m$  passes. The proof is complete. ■

There is a relation between the number of periodic curves and that of conditionally isolated periodic points, as is stated in the following lemma.

**Lemma 7.7** *If  $f$  has at most finitely many irreducible periodic curves or equivalently if the set  $P(f)$  is finite (see Remark 7.1), then the set  $\text{Per}^{ci}(f)$  is also finite.*

*Proof.* First we show that any conditionally isolated periodic point  $x \in \text{Per}^{ci}(f)$  is an intersection point of two or more distinct irreducible periodic curves of  $f$ . Let  $n$  and  $m$  be the prime period and the secondary period of  $x$  relative to  $f$ , respectively. By Remark 7.4 there exists an integer  $k \geq 2$  such that  $m = kn$ . Let  $C$  be an irreducible fixed curve of  $f^m$  passing through  $x$ . Since  $x \in X_0(f^n)$ , one has either  $x \in X_0^\circ(f^n)$  or  $x \in X_0^\circ(f^{-n})$  (see Definition 3.3). In the former case, for each  $1 \leq l \leq k-1$ , let  $C_l := f^{ln}(C)$  be the strict transform of  $C$  by  $f^{ln}$ . In the latter case we consider  $C_l := f^{-ln}(C)$  instead. Then  $C_l$  is an irreducible fixed curve of  $f^m$  passing through  $x$ , but different from  $C$ . Thus  $x \in C \cap C_l$  is an intersection point of the distinct irreducible periodic curves  $C$  and  $C_l$  of  $f$ . So the claim is verified.

We proceed to the proof of the lemma. Assume that  $f$  has at most finitely many irreducible periodic curves. Then the set of all intersection points of all pairs of distinct irreducible periodic curves is also finite. By what is proved in the last paragraph,  $\text{Per}^{ci}(f)$  is a subset of this finite set. Hence the set  $\text{Per}^{ci}(f)$  is also finite. ■

Formula (37) shows that the weighted cardinalities  $\#\text{Per}_n^i(f)$  are controlled by the magnitudes of the Lefschetz numbers  $L(f^n)$  and the sets  $P_n(f)$  of prime periods of periodic curves. In terms of the first dynamical degree  $\lambda(f)$  the behavior of  $L(f^n)$  is described as follows.

**Lemma 7.8** *If  $f : X \rightarrow X$  is an AS bimeromorphic map on a compact Kähler surface  $X$  with the first dynamical degree  $\lambda(f) > 1$ , then the Lefschetz numbers  $L(f^n)$  admits the estimate:*

$$|L(f^n) - \lambda(f)^n| \leq \begin{cases} O(1) & (\text{if } X \sim \text{no complex 2-torus}), \\ 4\lambda(f)^{n/2} + O(1) & (\text{if } X \sim \text{a complex 2-torus}), \end{cases}$$

where  $X \sim Y$  indicates that  $X$  is bimeromorphically equivalent to  $Y$ .

*Proof.* In what follows we use the notation:  $h^i := \dim_{\mathbb{C}} H^i(X)$ ,  $h^{i,j} := \dim_{\mathbb{C}} H^{i,j}(X)$  and  $t_n^{i,j} := \text{Tr}[(f^n)^* : H^{i,j}(X) \rightarrow H^{i,j}(X)]$ . Note that  $L(f^n) = \sum_{i,j} (-1)^{i+j} t_n^{i,j}$  and  $t_n^{i,j}$  is the  $n$ -th power sum of the eigenvalues of  $f^* : H^{i,j}(X) \rightarrow H^{i,j}(X)$ . First, for  $i = 0, 2$ , the induced map  $(f^n)^* : H^{i,i}(X) \cong \mathbb{C} \rightarrow \mathbb{C}$  is the identity since  $f^n$  is a bimeromorphic map. Next we consider the case  $(i, j) = (1, 1)$ . Because  $f$  is assumed to be AS, we have  $(f^n)^* = (f^*)^n : H^{1,1}(X) \rightarrow H^{1,1}(X)$ . It follows from [5, Theorem 0.3] that  $f^*|_{H^{1,1}(X)}$  has a simple eigenvalue  $\lambda(f) > 1$  together with all the remaining eigenvalues in the closed unit disk in  $\mathbb{C}$ . This shows that

$$t_n^{1,1} = \lambda(f)^n + O(1). \quad (39)$$

It is well known that the first dynamical degree  $\lambda(f)$  and the eigenvalues of  $(f^n)^* : H^{i,j}(X) \rightarrow H^{i,j}(X)$  with  $(i, j) \neq (1, 1)$  are invariant under meromorphic conjugation of  $(X, f)$  (see [5, Proposition 1.18] and [2, page 34]). Hence for any  $(i, j)$  the number  $t_n^{i,j}$  is also invariant except for the  $O(1)$ -term of (39) in the case  $(i, j) = (1, 1)$ . So, in any case, every  $t_n^{i,j}$  is invariant up to  $O(1)$ -term. Thus we may assume from the beginning that  $(X, f)$  is of the canonical form (3) or (4) in Theorem 5.2. We make a case-by-case check. If  $X$  is either a rational surface or an Enriques surface, then  $h^1 = h^{2,0} = h^{0,2} = h^3 = 0$  and so (39) implies that

$$L(f^n) = t_n^{1,1} + 2 = \lambda(f)^n + O(1). \quad (40)$$

Next consider the case where  $f$  is an automorphism of a K3 surface  $X$ . Then we have  $h^1 = h^3 = 0$  and  $H^{2,0}(X) = \mathbb{C}\eta$  and  $H^{0,2}(X) = \mathbb{C}\bar{\eta}$ , where  $\eta$  is a nowhere vanishing holomorphic 2-form on  $X$ . There exists a constant  $\delta \in \mathbb{C}^\times$  such that  $f^*|_{H^{2,0}(X)}$  and  $f^*|_{H^{0,2}(X)}$  are the scalar multiplications by  $\delta$  and by  $\bar{\delta}$  respectively. Since the automorphism  $f$  preserves the volume form  $\eta \wedge \bar{\eta}$ , we have  $\eta \wedge \bar{\eta} = f^*\eta \wedge \bar{\eta} = (\delta\eta) \wedge (\bar{\delta}\bar{\eta}) = |\delta|^2\eta \wedge \bar{\eta}$  and  $|\delta| = 1$ . Again (39) yields:

$$L(f^n) = t_n^{1,1} + \delta^n + \bar{\delta}^n + 2 = \lambda(f)^n + O(1). \quad (41)$$

Finally we treat the case where  $f$  is an automorphism of a complex 2-torus  $X = \mathbb{C}^2/\Gamma$  with  $\Gamma \cong \mathbb{Z}^4$  being a lattice in  $\mathbb{C}^2$ . The map  $f$  lifts to an affine automorphism

$$F : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (z_1, z_2) \mapsto (a_{11}z_1 + a_{12}z_2 + b_1, a_{21}z_1 + a_{22}z_2 + b_2)$$

through the canonical projection  $\mathbb{C} \rightarrow \mathbb{C}/\Gamma$ . The determinant of the matrix  $A := (a_{ij}) \in M_2(\mathbb{C})$  satisfies  $|\det A| = 1$ , since  $F$  acts on  $\Gamma$  bijectively. Let  $\delta_1$  and  $\delta_2$  be the eigenvalues of  $A$  with  $|\delta_1| \geq |\delta_2|$ . Note that  $|\det A| = 1$  implies  $|\delta_1\delta_2| = 1$ . If we put  $\delta := \delta_1$  and  $\varepsilon := \delta_1\delta_2$ , then  $|\delta| \geq 1$  and  $|\varepsilon| = 1$ . The action  $f^* : H^{1,0}(X) = \mathbb{C}dz_1 \oplus \mathbb{C}dz_2 \rightarrow \mathbb{C}dz_1 \oplus \mathbb{C}dz_2$  is represented by the matrix  $A$  and so has the eigenvalues  $\delta_1 = \delta$  and  $\delta_2 = \varepsilon\delta^{-1}$ . In a similar manner, using the representations

$$H^{i,j}(X) = \bigoplus_{k_1 < \dots < k_i} \bigoplus_{l_1 < \dots < l_j} \mathbb{C} dz_{k_1} \wedge \dots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_j},$$



$(i, j)$	the eigenvalues of $f^* : H^{i,j}(X) \hookrightarrow H^{i,j}(X)$
$(1, 0)$	$\delta_1 = \delta, \quad \delta_2 = \varepsilon\delta^{-1}$
$(0, 1)$	$\bar{\delta}_1 = \bar{\delta}, \quad \bar{\delta}_2 = \bar{\varepsilon}\bar{\delta}^{-1}$
$(2, 0)$	$\delta_1\delta_2 = \varepsilon$
$(0, 2)$	$\bar{\delta}_1\bar{\delta}_2 = \bar{\varepsilon}$
$(2, 1)$	$\delta_1\delta_2\bar{\delta}_1 = \varepsilon\bar{\delta}, \quad \delta_1\delta_2\bar{\delta}_2 = \bar{\delta}^{-1}$
$(1, 2)$	$\delta_1\bar{\delta}_1\bar{\delta}_2 = \bar{\varepsilon}\delta, \quad \delta_2\bar{\delta}_1\bar{\delta}_2 = \delta^{-1}$
$(1, 1)$	$ \delta_1 ^2 =  \delta ^2, \quad \delta_1\bar{\delta}_2 = \bar{\varepsilon}(\delta/\bar{\delta}), \quad \delta_2\bar{\delta}_1 = \varepsilon(\bar{\delta}/\delta), \quad  \delta_2 ^2 =  \delta ^{-2}$

Table 1: The eigenvalues of  $f^* : H^{i,j}(X) \hookrightarrow H^{i,j}(X)$  for a complex 2-torus  $X$

the eigenvalues of  $f^* : H^{i,j}(X) \hookrightarrow H^{i,j}(X)$  are given as in Table 1. So the Lefschetz numbers  $L(f^n) = \sum_{i,j} (-1)^{i+j} t_n^{i,j}$  are calculated as

$$L(f^n) = |\delta|^{2n} + |\delta|^{-2n} + 2 - 2 \operatorname{Re} \{ (1 + \bar{\varepsilon}^n)\delta^n + (1 + \varepsilon^n)\delta^{-n} - \varepsilon^n(1 + (\bar{\delta}/\delta)^n) \}.$$

Since  $|\delta| \geq 1$ , the spectral radius of  $f^*|_{H^{1,1}(X)}$  is given by  $|\delta|^2$  and we have  $\lambda(f) = |\delta|^2 > 1$ . Using this relation in the formula above, we can easily obtain the estimate:

$$|L(f^n) - \lambda(f)^n| < 4\lambda(f)^{n/2} + 11 \quad (n \in \mathbb{N}). \quad (42)$$

Now the lemma follows from the estimates (40), (41) and (42), where the constant 11 in (42) should be replaced by  $O(1)$  if  $X$  is a proper modification of a 2-torus. ■

With these preliminaries we establish the following theorem.

**Theorem 7.9** *Let  $X$  be a smooth projective surface and  $f : X \rightarrow X$  an AS birational map without periodic curves of type I. If  $\lambda(f) > 1$  then  $f$  has at most finitely many irreducible periodic curves, at most finitely many conditionally isolated periodic points, and infinitely many absolutely isolated periodic points. Moreover we have  $\#\operatorname{Per}_n^{ci}(f) = O(1)$  and*

$$|\#\operatorname{Per}_n^{ai}(f) - \lambda(f)^n| \leq \begin{cases} O(1) & (\text{if } X \sim \text{no Abelian surface}), \\ 4\lambda(f)^{n/2} + O(1) & (\text{if } X \sim \text{an Abelian surface}), \end{cases} \quad (43)$$

where  $X \sim Y$  indicates that  $X$  is birationally equivalent to  $Y$ .

*Proof.* By the assumption that  $f$  has no periodic curves of type I, any irreducible periodic curve of it is of type II. Since  $\lambda(f) > 1$ , the assertion (1) of Theorem 2.4 implies that  $f$  has at most finitely many irreducible periodic curves, namely, that  $\operatorname{Card} P(f) < \infty$ . Since  $P_n(f)$  is a subset of  $P(f)$  for any  $n \in \mathbb{N}$ , the second term of the righthand side of formula (37) is bounded as a function of  $n \in \mathbb{N}$ . So the Lefschetz numbers  $L(f^n)$  and the weighted cardinalities  $\#\operatorname{Per}_n^i(f)$  behave in the same manner modulo a bounded function of  $n$ :

$$\#\operatorname{Per}_n^i(f) = L(f^n) + O(1). \quad (44)$$

For each  $x \in \text{Per}^i(f)$  let  $n_x$  denote the prime period of  $x$  relative to  $f$ . By Lemma 7.6 there exists a constant  $M_x < \infty$  such that  $\nu_x(f^{l \cdot n_x}) \leq M_x$  for all  $l \in \mathbb{N}$ . Since  $P(f)$  is finite, Lemma 7.7 implies that  $\text{Per}^{ci}(f)$  is also finite. As  $n/n_x \in \mathbb{N}$  for every  $x \in \text{Per}_n^i(f)$ , we have

$$\begin{aligned} \#\text{Per}_n^{ci}(f) &:= \sum_{x \in \text{Per}_n^{ci}(f)} \nu_x(f^n) = \sum_{x \in \text{Per}_n^{ci}(f)} \nu_x(f^{(n/n_x) \cdot n_x}) \\ &\leq \sum_{x \in \text{Per}_n^{ci}(f)} M_x \leq \sum_{x \in \text{Per}^{ci}(f)} M_x < \infty, \end{aligned}$$

which leads to  $\#\text{Per}_n^{ci}(f) = O(1)$ . Then this together with (44) yields

$$\#\text{Per}_n^{ai}(f) = L(f^n) + O(1). \quad (45)$$

We show that the set  $\text{Per}^{ai}(f)$  is infinite. Assume the contrary that it is finite. Then the same estimate as above with  $\text{Per}_n^{ci}(f)$  replaced by  $\text{Per}_n^{ai}(f)$  yields

$$\#\text{Per}_n^{ai}(f) \leq \sum_{x \in \text{Per}^{ai}(f)} M_x < \infty.$$

This estimate and formula (45) imply that the Lefschetz numbers  $L(f^n)$  are bounded, but this contradicts Lemma 7.8. Finally, formula (43) follows from Lemma 7.8 and formula (45). ■

*Proof of Theorem 2.8.* By Theorem 2.6 and condition (\*), the map  $f$  has no periodic curves of type I. Then Theorem 2.8 is an immediate consequence of Theorem 7.9. ■

There is a counterpart of Theorem 7.9 for the case  $\lambda(f) = 1$ .

**Proposition 7.10** *Let  $f : X \rightarrow X$  be an AS birational map without periodic curves of type I. If  $\lambda(f) = 1$  and the Lefschetz numbers  $L(f^n)$  are unbounded, then either*

- (1)  *$f$  has at most finitely many irreducible periodic curves, at most finitely many conditionally isolated periodic points, and infinitely many absolutely isolated periodic points; or*
- (2)  *$f$  has infinitely many irreducible periodic curves and preserves a unique rational or elliptic fibration such that any irreducible periodic curve is contained in a fiber of the fibration.*

*Proof.* Since the Lefschetz numbers  $L(f^n)$  are unbounded,  $f^n$  is not isotopic to the identity for any  $n \in \mathbb{N}$ . Thus we are in case (1) or (2) of Theorem 5.2, so that  $f$  preserves a unique rational or elliptic fibration. The remaining proof is similar to the proof of Theorem 7.9, again making use of Theorem 7.2, Lemmas 7.6 and 7.7. The difference is to apply assertion (2) of Theorem 2.4 instead of assertion (1) of the same theorem, and to apply Theorem 5.10 instead of Theorems 5.3 and 5.5. Details may be omitted. ■

## 8 An Example

In order to illustrate our main theorems, we give an example of an AS birational map preserving a meromorphic 2-form on a smooth projective rational surface. This example arises as a special case of a 4-parameter family of dynamical systems on cubic surfaces derived from the nonlinear monodromy of the sixth Painlevé equation via the Riemann-Hilbert correspondence [10, 11].

Let  $\overline{S}$  be the projective cubic surface in  $\mathbb{P}^3$  defined by the homogeneous cubic equation:

$$Z_1 Z_2 Z_3 + Z_0(Z_1^2 + Z_2^2 + Z_3^2) - 8Z_0^2(Z_1 + Z_2 + Z_3) + 28Z_0^3 = 0,$$

in homogeneous coordinates  $Z = [Z_0 : Z_1 : Z_2 : Z_3]$ . It has a unique singularity at

$$q = [1 : 2 : 2 : 2],$$

which turns out to be a simple singularity of type  $D_4$ . The intersection of  $\overline{S}$  with the plane  $\{Z_0 = 0\}$  at infinity yields tritangent lines  $L_i = \{Z_0 = Z_i = 0\}$  at infinity ( $i = 1, 2, 3$ ). Then the affine cubic surface  $S := \overline{S} \setminus L$  with  $L := L_1 \cup L_2 \cup L_3$  is given by the affine cubic equation:

$$g(z) := z_1 z_2 z_3 + z_1^2 + z_2^2 + z_3^2 - 8(z_1 + z_2 + z_3) + 28 = 0,$$

where  $z_i := Z_i/Z_0$ . Since this equation is quadratic in each variable  $z_i$ , the line through a point  $z \in S$  parallel to the  $z_i$ -axis passes through a unique second point  $\sigma_i(z) \in S$ . Hence we have three involutive automorphisms  $\sigma_i : S \rightarrow S$ , which are written explicitly as

$$\sigma_i : (z_i, z_j, z_k) \mapsto (8 - z_i - z_j z_k, z_j, z_k) \quad (i = 1, 2, 3),$$

with  $\{i, j, k\} = \{1, 2, 3\}$ . Note that the singular point  $q$  is a fixed point of the involutions  $\sigma_i$ .

A natural (complex) area-form on  $S$  is given by its Poincaré residue:

$$\omega_S := \frac{dz_1 \wedge dz_2 \wedge dz_3}{dg} \quad \text{restricted on } S \setminus \{q\}.$$

The map  $\sigma_i$  sends  $\omega_S$  to its negative:  $\sigma_i^* \omega_S = -\omega_S$ . Moreover  $\sigma_i$  extends to a birational map  $\overline{\sigma}_i : \overline{S} \rightarrow \overline{S}$  and  $\omega_S$  extends to a 2-form  $\omega_{\overline{S}}$  which is holomorphic on  $S \setminus \{q\}$  and meromorphic on  $\overline{S} \setminus \{q\}$  with simple poles along the tritangent lines at infinity:  $(\omega_{\overline{S}})_\infty = L_1 + L_2 + L_3$ . We consider an  $\omega_{\overline{S}}$ -preserving birational map defined by

$$\overline{\sigma} := (\overline{\sigma}_1 \circ \overline{\sigma}_2 \circ \overline{\sigma}_3)^2 : \overline{S} \rightarrow \overline{S}.$$

For the same reason as in [11, Lemma 16],  $\overline{\sigma}$  has no periodic curves of any prime period.

Let  $\pi : (X, E) \rightarrow (\overline{S}, q)$  be a minimal desingularization of  $\overline{S}$ . Then its exceptional set  $E$  consists of four irreducible components  $E_0, E_1, E_2, E_3$  as depicted in Figure 1. We denote the strict transform of  $L_i$  by the same symbol  $L_i$ . Then the pull-back  $\omega_X := \pi^* \omega_{\overline{S}}$  turns out to be a meromorphic 2-form on  $X$  with simple poles along the tritangent lines at infinity:

$$(\omega_X)_\infty = L := L_1 + L_2 + L_3.$$

Note that  $\omega_X$  is holomorphic and nondegenerate on  $X \setminus L$  even around  $E$ . Let  $p_i$  be the intersection point of  $L_j$  and  $L_k$  for  $\{i, j, k\} = \{1, 2, 3\}$ . It is easy to see that the map  $\overline{\sigma} : \overline{S} \rightarrow \overline{S}$  lifts to a birational map  $f : X \rightarrow X$  such that  $f^{-n} I(f) = \{p_3\}$  and  $f^n I(f^{-1}) = \{p_1\}$  for any  $n \in \mathbb{N}$  (see [11, formula (52)]). Hence  $f$  is AS. We observe that  $E_0, E_1, E_2$  and  $E_3$  are irreducible fixed curves of  $f$ . There is no other periodic curves of  $f$ , since  $\overline{\sigma}$  has no periodic curves as mentioned earlier. Thus under the notation of Section 6 we have

$$P(f) = \{1\}, \quad \text{PC}_1(f) = \{E_0, E_1, E_2, E_3\}, \quad C_1(f) = E := E_0 \cup E_1 \cup E_2 \cup E_3. \quad (46)$$

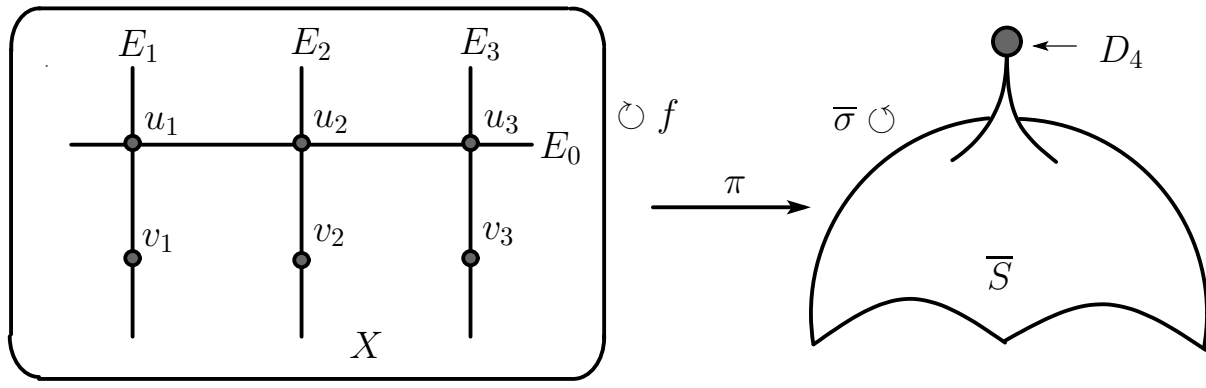


Figure 1: Minimal resolution of singularities of type  $D_4$

Note that  $f$  preserves the meromorphic 2-form  $\omega_X$ . Since  $\omega_X$  is holomorphic in a neighborhood of  $E$ , Theorem 2.6 implies that each fixed curve  $E_i$  is of type II relative to  $f$ . We wish to calculate the number  $\xi_1(f)$  defined in (36).

Let  $u_i$  denote the intersection point of  $E_0$  and  $E_i$  ( $i = 1, 2, 3$ ). We calculate the indices  $\nu_{u_i}(f)$  and  $\nu_{E_j}(f)$  ( $j = 0, 1, 2, 3$ ). The induced endomorphism  $f_{u_i}^* : A_{u_i} \rightarrow A_{u_i}$  is represented as

$$f_{u_i}^*(z_1) = z_1 + z_1^3 z_2 h_1(z), \quad f_{u_i}^*(z_2) = z_2 + z_1^2 z_2^2 h_2(z),$$

with some units  $h_1, h_2 \in A_{u_i}^\times$ , where  $z_1$  and  $z_2$  are local coordinates around  $u_i$  such that  $E_0 = \{z_1 = 0\}$  and  $E_i = \{z_2 = 0\}$ . By the definitions of  $\mathbf{a}(\sigma)$  and  $\mathbf{b}(\sigma)$  in Section 3 and (11),

$$\begin{aligned} \mathbf{a}(f_{u_i}^*) &= (z_1^2 z_2), \\ \mathbf{b}(f_{u_i}^*) &= (z_1 h_1(z), z_2 h_2(z)) = (z_1, z_2), \\ \varpi_{f_{u_i}^*} &= z_2 h_2(z) \cdot dz_1 - z_1 h_1(z) \cdot dz_2 \in \hat{\Omega}_{A_{u_i}/\mathbb{C}}^1, \end{aligned}$$

from which (9), (10), (13) and (20) yield

$$\begin{aligned} \delta(f_{u_i}^*) &= \dim_{\mathbb{C}} A_{u_i}/(z_1, z_2) = 1, \\ \nu_{(z_1)}(f_{u_i}^*) &= \nu_{E_0}(f) = 2, \\ \nu_{(z_2)}(f_{u_i}^*) &= \nu_{E_i}(f) = 1, \\ \mu_{(z_1)}(f_{u_i}^*) &= \mu_{(z_2)}(f_{u_i}^*) = 1. \end{aligned}$$

By substituting these results into (14) and (19), the index of  $f$  at  $u_i$  is given by

$$\nu_{u_i}(f) = \delta(f_{u_i}^*) + \sum_{k=1}^2 \nu_{(z_k)}(f_{u_i}^*) \cdot \mu_{(z_k)}(f_{u_i}^*) = 4.$$

An extra work shows that besides  $u_1, u_2, u_3$ , there are exactly three other points  $x \in E$  such that  $\nu_x(f) > 0$ . More precisely, for each  $i = 1, 2, 3$ , there is a unique such point  $v_i \in E_i \setminus \{u_i\}$ ,

at which one has  $\nu_{v_i}(f) = 2$  (see Figure 1). Summarizing these calculations, we have

$$\nu_x(f) = \begin{cases} 4 & (x = u_i, i = 1, 2, 3), \\ 2 & (x = v_i, i = 1, 2, 3), \\ 0 & (x : \text{any other point on } E), \end{cases} \quad (47)$$

$$\nu_{E_i}(f) = \begin{cases} 2 & (i = 0), \\ 1 & (i = 1, 2, 3). \end{cases} \quad (48)$$

Since  $\tau_{E_i} = -2$  for each  $i \in \{0, 1, 2, 3\}$ , substituting (46), (47) and (48) into (36) yields

$$\xi_1(f) := \sum_{x \in C_1(f)} \nu_x(f) + \sum_{C \in \text{PC}_1(f)} \tau_C \cdot \nu_C(f) = \sum_{x \in E} \nu_x(f) + \sum_{i=0}^3 \tau_{E_i} \cdot \nu_{E_i}(f) = 8.$$

For every  $n \in \mathbb{N}$  one has  $P_n(f) = P(f) = \{1\}$  and thus the fixed point formula (37) yields

$$L(f^n) = \#\text{Per}_n^i(f) + \xi_1(f) = \#\text{Per}_n^i(f) + 8.$$

On the other hand, we are able to show that  $\lambda(f) = 9 + 4\sqrt{5}$  and  $L(f^n) = \lambda(f)^n + \lambda(f)^{-n} + 6$ . Therefore we arrive at the following explicit formula for the number of isolated periodic points:

$$\#\text{Per}_n^i(f) = (9 + 4\sqrt{5})^n + (9 + 4\sqrt{5})^{-n} - 2.$$

For this example all the isolated periodic points are absolutely isolated periodic points.

## References

- [1] M. Atiyah and R. Bott, *Lefschetz fixed point formula for elliptic complexes: I*, Ann. of Math. (2) **86** (1967), no. 2, 374–407; *II. Applications*, ibid. **88** (1968), no. 3, 451–491.
- [2] W. Barth, K. Hulek, C. Peters and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, 2004.
- [3] M. Brown, *On the fixed point index of iterates of planar homeomorphisms*, Proc. Amer. Math. Soc. **108** (1990), 1109–1114.
- [4] S. Cantat and C. Favre, *Symétries birationnelles des surfaces feuilletées*, J. Reine Angew. Math. **561** (2003), 199–235.
- [5] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), no. 6, 1135–1169.
- [6] J. Diller, D. Jackson and A. Sommese, *Invariant curves for birational surface maps*, Trans. Amer. Math. Soc. **359** (2007), no. 6, 2973–2991.
- [7] N. Fagella and J. Llibre, *Periodic points of holomorphic maps via Lefschetz numbers*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4711–4730.

- [8] C. Favre, *Points périodiques d'applications birationnelles de  $\mathbb{P}^2$* , Ann. Inst. Fourier. **48** (1998), no. 4, 999–1023.
- [9] S. Iitaka, *Algebraic geometry*, Springer-Verlag, New York, Berlin, 1982.
- [10] M. Inaba, K. Iwasaki and M.-H. Saito, *Dynamics of the sixth Painlevé equation*, Théories asymptotiques et équations de Painlevé, Séminaires et Congrès **14** (2006), 103–167.
- [11] K. Iwasaki and T. Uehara, *An ergodic study of Painlevé VI*, Math. Ann. **338** (2007), no. 2, 295–345.
- [12] Y. Matsui and K. Takeuchi, *Microlocal study of Lefschetz fixed point formulas for higher-dimensional fixed point sets*, preprint.
- [13] V.K. Patodi, *Holomorphic Lefschetz fixed point formula*, Bull. Amer. Math. Soc. **79** (1973), 825–828.
- [14] S. Saito, *General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings*, Amer. J. Math. **109** (1987), no. 6, 1009–1042.
- [15] M. Shub and D. Sullivan, *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.
- [16] D. Toledo and Y. L. Tong, *Duality and intersection theory in complex manifolds. II. The holomorphic Lefschetz formula*, Ann. of Math. (2) **108** (1978), no. 3, 519–538.
- [17] G.Y. Zhang, *Fixed point indices and periodic points of holomorphic mappings*, Math. Ann. **337** (2007), no. 2, 401–433.